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CONCEPT MAP

CLASS XI-XII

MATHEMATICS today

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Corporate Office:
Plot 99, Sector 44 Institutional Area,
Gurugram -122 003 (HR), Tel : 0124-6601200
e-mail : info@mtg.in website : www.mtg.in

Regd. Office:
406, Taj Apartment, Near Safdarjung Hospital,
Ring Road, New Delhi - 110029.
Managing Editor : Mahabir Singh
Editor : Anil Ahlawat

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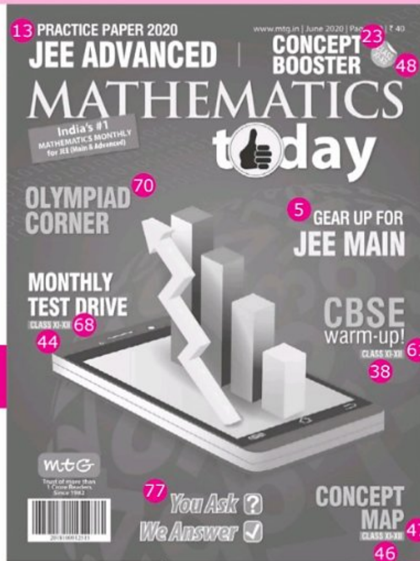

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GEAR UP FOR JEE MAIN

EXAM ON 18th-23rd July 2020

with Numerical Value Type Questions

1. A letter is taken out at random from 'ASSISTANT' and another letter taken out from the letter of the word 'STATISTICS'. The probability that they are identical letters, is

(a) $\frac{13}{90}$ (b) $\frac{1}{45}$

(c) $\frac{19}{90}$ (d) none of these

2. The value of $\tan i\left[\log_e\left(\frac{a-ib}{a+ib}\right)\right]=$

(a) 0 (b) 1

(c) -1 (d) none of these

3. Let $A=\begin{bmatrix}0 & 2b & c\\ a & b & -c\\ a & -b & c\end{bmatrix}$ be an orthogonal matrix the values of a, b, c are related with

(a) $a=\pm\frac{1}{\sqrt{2}}, b=\pm\frac{1}{\sqrt{6}}, c=\pm\frac{1}{\sqrt{3}}$

(b) $a=\pm\frac{1}{\sqrt{2}}, c=\pm\frac{1}{\sqrt{6}}$

(c) $a=\pm\frac{1}{\sqrt{2}}, b=\pm\frac{1}{\sqrt{6}}$

(d) none of these

4. If maximum and minimum values of the determinant $\begin{vmatrix}1+\cos^2 x & \sin^2 x & \cos 2x\\ \cos^2 x & 1+\sin^2 x & \cos 2x\\ \cos^2 x & \sin^2 x & 1+\cos 2x\end{vmatrix}$ are α and β respectively then

(a) $\alpha^2+\beta^{101}=10$

(b) $\alpha^3-\beta^{99}=26$

(c) $2\alpha^2-18\beta^{11}=0$

(d) all of these

5. If $\int\frac{dx}{4\sin^2 x+4\sin x\cos x+5\cos^2 x}$ $=A\tan^{-1}(B\tan x+C)$ then

(a) $A=\frac{1}{4}, B=\frac{1}{2}, C=1$

(b) $A=\frac{1}{2}, B=\frac{1}{4}, C=1$

(c) $A=1, B=\frac{1}{2}, C=\frac{1}{4}$

(d) $A=\frac{1}{4}, B=1, C=\frac{1}{2}$

6. If $I(x)=\int_0^{\pi/2}\frac{dx}{\sqrt{1+\sin^3 x}}$ and $a\le I(x)\le b$, then

(a) $a=\pi\sqrt{2}, b=\frac{\pi}{2}$ (b) $a=\frac{\pi}{2\sqrt{2}}, b=\frac{\pi}{4}$

(c) $a=\frac{\pi}{2\sqrt{2}}, b=\frac{\pi}{2}$ (d) $a=\frac{\pi}{\sqrt{2}}, b=\pi$

7. The area of the parallelogram whose sides are along the straight lines $y = 3x + 5$, $y = 3x + 2$, $y = 5x + 4$ and $y = 5x - 1$ equals

(a) $\frac{15}{2}$ sq. units (b) 15 sq. units

(c) $\frac{15}{\sqrt{10}\sqrt{26}}$ sq. units

(d) none of these

8. Tangent drawn from the point (4, 4) to the circle $x^2 + y^2 - 2x - 2y - 7 = 0$ meet the circle at A and B. The length of the chord AB equals

(a) $4\sqrt{3}$ (b) $2\sqrt{3}$ (c) $2\sqrt{6}$ (d) $3\sqrt{2}$

9. The set of values of m for which a chord of slope m of the circle $x^2 + y^2 = 16$ touches the parabola $y^2 = 8x$

(a) $(-\infty, -1) \cup (1, \infty)$
(b) $(-\infty, \infty)$
(c) $\left(-\infty, -\sqrt{\frac{\sqrt{2}-1}{2}}\right) \cup \left(\sqrt{\frac{\sqrt{2}-1}{2}}, \infty\right)$
(d) none of these

10. If the normal at an end of a latus rectum passes through the extremity of the minor axis then eccentricity of the ellipse is given by
(a) $e^2 + e + 1 = 0$ (b) $e^4 + e^2 + 1 = 0$
(c) $e^4 - e^2 - 1 = 0$ (d) $e^4 + e^2 - 1 = 0$

11. If the $(m + 1)^{th}$, $(n + 1)^{th}$, and $(r + 1)^{th}$ term of an A.P. are in G.P. and m, n, r are in H.P., then the ratio of the first term of the A.P. to its common difference is
(a) $n/2$ (b) $-n/2$
(c) $n/4$ (d) $-2n$

12. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x - 1)$. If the graph passes through the point (2, 1) and at that point the tangent to the curve is $y = 3x - 5$, then the function is
(a) $(x + 1)^3$ (b) $(x - 1)^3$
(c) $(x - 1)^2$ (d) $(x + 1)^2$

13. The function $f(x)=\begin{cases}\frac{\sin^3 x^2}{x} & \forall x \neq 0 \\ 0 & \forall x = 0\end{cases}$
(a) is continuous but not derivable at $x = 0$
(b) neither continuous nor derivable at $x = 0$
(c) continuous and differentiable at $x = 0$
(d) none of these.

14. If $f(x)=\int_{x^2}^{x^3} \frac{dt}{\log t}, x>0$ then
(a) $f(x)$ is maximum at $x = 1$
(b) $f(x)$ is an increasing function $\forall\, x \in R^+$ only
(c) $f(x)$ is minimum at $x = 1$
(d) $f(x)$ is neither maximum nor minimum at $x = 1$ but an increasing function for all x belong to R

15. If A_1 and A_2 respectively represents the area bounded by the curves $f(x, y) : 4x^2 \le y \le 3x$ and $g(x, y) : 4x^2 \le y \le |3x|$ then $A_1 : A_2$ equals
(a) 2 : 1 (b) 3 : 1 (c) 1 : 2 (d) 1 : 3

16. Solution of differential equation $\frac{dt}{dx}=\frac{t\left(\frac{dg(x)}{dx}\right)-t^2}{g(x)}$ is
(a) $t=\frac{g(x)+c}{x}$ (b) $t=\frac{g(x)}{x}+c$
(c) $t=\frac{g(x)}{x+c}$ (d) $t = g(x) + x + c$

17. Let Y be a set of all complex numbers such that $|z| = 1$ and defined relation R on Y by z_1Rz_2 is $|\arg z_1-\arg z_2|=\frac{2\pi}{3}$ then R is
(a) symmetric (b) transitive
(c) anti-symmetric (d) reflexive

18. If $\lim_{x\to 0}\frac{f(x)}{x}=2$ where $f(x)=\min.\left\{\sin\sqrt{[m]}x, |x|\right\}$ and $[\cdot]$ greatest integer then
(a) $m \in \{4\}$ (b) $m \in [4, 5]$
(c) $m \in [4, 5)$ (d) $m = \{5\}$

19. The angle between the line $\vec{r}=(2\hat{i}+2\hat{j}+\hat{k})+\lambda(2\hat{i}-3\hat{j}+2\hat{k})$ and the plane $\vec{r}\cdot(3\hat{i}-2\hat{j}+5\hat{k}) = 4$
(a) $\sin^{-1}\left(\frac{11}{\sqrt{323}}\right)$ (b) $\sin^{-1}\left(\frac{22}{\sqrt{323}}\right)$
(c) $\sin^{-1}\left(\frac{22}{\sqrt{646}}\right)$ (d) none of these

20. The line $\frac{x-3}{2}=\frac{y-4}{5}=\frac{z-6}{7}$
(a) lies in $3x + 2y + 4z - 6 = 0$
(b) is parallel to $2x - 5y + 3z = 9$
(c) is perpendicular to $2x - 5y + 3z = 0$
(d) passing through (1, 2, 3)

NUMERICAL VALUE TYPE

21. The number of ways of arranging the word 'ARRANGE' so that neither 2A's nor 2R's occurs together are

22. A man has seven relatives 4 of them are ladies and 3 gentlemen, his wife has 7 relatives out of which 3 are ladies and 4 gentlemen. The number of ways in which they can invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of the man's relative and 3 of the wife's relative are

23. If $\log_{\sqrt{7}} x + \log_{\sqrt[3]{7}} x + \log_{\sqrt[4]{7}} x + \ldots$ upto 20^{th} term $= 230$ then the value of x equals

24. If α, β are roots of the equation $p(x^2 - x) + x + 5 = 0$ and p_1, p_2 are two values of p for which the roots α, β are connected by the relation $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$ then the value of $\frac{p_1}{p_2} + \frac{p_2}{p_1} =$

25. The value of $3\tan^6 10° - 27\tan^4 10° + 33\tan^2 10°$ is

SOLUTIONS

1. (c) : In the word 'ASSISTANT' there are SSS AA I TT N and in 'STATISTICS" there are A C II SSS TTT. The non common letters are N and C. The identical letters are A, I, S, T.

Probability of getting $A = \frac{^2C_1}{^9C_1} \times \frac{^1C_1}{^{10}C_1} = \frac{1}{45}$

Probability of getting $I = \frac{^1C_1}{^9C_1} \times \frac{^2C_1}{^{10}C_1} = \frac{1}{45}$

Probability of getting $S = \frac{^3C_1}{^9C_1} \times \frac{^3C_1}{^{10}C_1} = \frac{1}{10}$

Probability of getting $T = \frac{^2C_1}{^9C_1} \times \frac{^3C_1}{^{10}C_1} = \frac{1}{15}$

$\therefore$ Required probability $= \frac{1}{45} + \frac{1}{45} + \frac{1}{10} + \frac{1}{15} = \frac{19}{90}$

2. (d) : $\tan i \left[\log_e \left(\frac{a - ib}{a + ib} \right) \right]$

$= \tan i \left[\log_e \left(\frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta} \right) \right]$

(putting $a = r\cos\theta$ and $b = r\sin\theta$)

$= \tan[i \log(\cos 2\theta - i \sin 2\theta)] = \tan(i \log_e e^{-2i\theta})$

$= \tan(-i^2 2\theta) = \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2ab}{a^2 - b^2}$

3. (a) : We have,

$$A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \therefore A^t = \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$$

As A is an orthogonal matrix, therefore $AA^t = I$

$$\Rightarrow \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4b^2 + c^2 & 2b^2 - c^2 & -2b^2 + c^2 \\ 2b^2 - c^2 & a^2 + b^2 + c^2 & a^2 - b^2 - c^2 \\ -2b^2 + c^2 & a^2 - b^2 - c^2 & a^2 + b^2 + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, from the definition of equality of two matrices we have, $4b^2 + c^2 = 1$, $2b^2 - c^2 = 0$, $a^2 + b^2 + c^2 = 1$

On solving the above equations we get,

$a = \pm\frac{1}{\sqrt{2}}, b = \pm\frac{1}{\sqrt{6}}, c = \pm\frac{1}{\sqrt{3}}$

4. (d) : Let $d = \begin{vmatrix} 1 + \cos^2 x & \sin^2 x & \cos 2x \\ \cos^2 x & 1 + \sin^2 x & \cos 2x \\ \cos^2 x & \sin^2 x & 1 + \cos 2x \end{vmatrix}$

Applying $R_1 \leftrightarrow R_1 - R_2$ and $R_2 \leftrightarrow R_2 - R_3$, we get

$d = 1(1 + \cos 2x + \sin^2 x) + 1(\cos^2 x) + 0$

$= 1 + \cos 2x + \sin^2 x + \cos^2 x = 2 + \cos 2x$

Now, we know that $-1 \le \cos 2x \le 1$

$\Rightarrow 2 - 1 \le 2 + \cos 2x \le 2 + 1 \Rightarrow 1 \le 2 + \cos 2x \le 3$

$\Rightarrow \beta \le 2 + \cos 2x \le \alpha \quad \therefore \alpha = 3, \beta = 1$

Now, $\alpha^2 + \beta^{101} = 9 + 1 = 10$

$\alpha^3 - \beta^{99} = 27 - 1 = 26$

$2\alpha^2 - 18\beta^{11} = 18 - 18 = 0$

5. (d) : Here $\int \frac{dx}{4\sin^2 x + 4\sin x \cos x + 5\cos^2 x}$

$= \int \frac{\sec^2 x \, dx}{4\tan^2 x + 4\tan x + 5} = \int \frac{dz}{(2z+1)^2 + 2^2}$, where $z = \tan x$

$=\frac{1}{2}\cdot\frac{1}{2}\tan^{-1}\left(\frac{2z+1}{2}\right)=\frac{1}{4}\tan^{-1}\left(\tan x+\frac{1}{2}\right)$

$=A\tan^{-1}(B\tan x+C)$

$\therefore A=\frac{1}{4}, B=1, C=\frac{1}{2}$

6. (c) : We have, $0\le x\le\frac{\pi}{2}$

$\Rightarrow \sin 0\le\sin x\le\sin\frac{\pi}{2} \Rightarrow 0\le\sin x\le 1$

$\Rightarrow 0\le\sin^3 x\le 1$

$\Rightarrow 1\le 1+\sin^3 x\le 2$

$\Rightarrow \frac{1}{2}\le\frac{1}{1+\sin^3 x}\le 1$

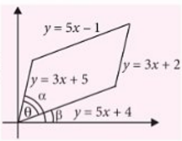
$\Rightarrow \int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{2}}dx\le\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1+\sin^3 x}}\le\int_0^{\frac{\pi}{2}}1dx$

$\Rightarrow \frac{\pi}{2\sqrt{2}}\le I(x)\le\frac{\pi}{2} \Rightarrow a=\frac{\pi}{2\sqrt{2}}, b=\frac{\pi}{2}$

7. (a) : Area of the parallelogram is $P_1P_2\text{cosec}\theta$, where P_1, P_2 are perpendicular distances between the two pair of parallel sides and θ be the angle between two adjacent sides.

Let the distance between the parallel lines $y = 5x + 4$ and $y = 5x - 1$ be P_1

$\therefore P_1=\frac{4-(-1)}{\sqrt{1^2+5^2}}=\frac{5}{\sqrt{26}}$



Also, let distance between the parallel lines $y = 3x + 5$ and $y = 3x + 2$ be P_2

$\therefore P_2=\frac{5-2}{\sqrt{1^2+3^2}}=\frac{3}{\sqrt{10}}$

Again, $\theta=\alpha-\beta, \tan\alpha=\frac{3}{1}, \tan\beta=\frac{5}{1}$

$\therefore \sin\theta=\sin(\alpha-\beta)=\cos\alpha\cos\beta(\tan\alpha-\tan\beta)$

$=(3-5)\times\frac{1}{\sqrt{10}}\cdot\frac{1}{\sqrt{26}}\quad\left[\because\cos\alpha=\frac{1}{\sqrt{10}},\cos\beta=\frac{1}{\sqrt{26}}\right]$

Therefore, $\text{cosec}\,\theta=\frac{\sqrt{10}\sqrt{26}}{2}$

(neglecting the negative sign).

$\therefore$ Required area $= P_1P_2\ \text{cosec}\theta$

$=\frac{5}{\sqrt{26}}\times\frac{3}{\sqrt{10}}\times\frac{\sqrt{10}\sqrt{26}}{2}=\frac{15}{2}$ sq.units.

8. (d) : The equation of circle is

$x^2 + y^2 - 2x - 2y - 7 = 0$

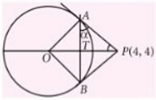
$\Rightarrow (x - 1)^2 + (y - 1)^2 = (3)^2.$

$\therefore$ Centre (1, 1) and radius = 3

Now, $PA=PB=\sqrt{16+16-8-8-7}=\sqrt{9}=3$

Again, $OP=(OA)^2+(AP)^2$

$=\sqrt{9+9}=3\sqrt{2}$



Now, in ΔATP, $\frac{AT}{AP}=\cos\alpha$

$\Rightarrow AT=AP\cos\alpha=3\cos 45°$

$\therefore AB=2AT=2\times 3\times\frac{1}{\sqrt{2}}=3\sqrt{2}$

9. (c) : For parabola $y^2 = 4ax$, the line $y = mx + c$ will be tangent if $c=\frac{a}{m}$.

$\therefore$ Equation of tangent to the parabola $y^2 = 8x$ is

$y=mx+\frac{2}{m}$

Now line to be chord of the circle $x^2 + y^2 = 16$ if distance from (0, 0) to the line will be less than the radius of the circle.

$\therefore \frac{\frac{2}{m}}{\sqrt{1+m^2}}<4 \Rightarrow \frac{2}{m}<4\sqrt{1+m^2}$

$\Rightarrow 1<4m^2(1+m^2) \Rightarrow 4m^4+4m^2-1>0$

$\Rightarrow m^4+m^2-\frac{1}{4}>0 \Rightarrow \left(m^2+\frac{1}{2}\right)^2-\frac{1}{2}>0$

$\left(\left(m^2+\frac{1}{2}\right)+\frac{1}{\sqrt{2}}\right)\left(\left(m^2+\frac{1}{2}\right)-\frac{1}{\sqrt{2}}\right)>0$

$m^2<-\frac{1}{2}+\frac{\sqrt{2}}{2}$ or $m^2>\frac{\sqrt{2}-1}{2}$

$\Rightarrow m\in\left(-\infty,-\sqrt{\frac{\sqrt{2}-1}{2}}\right)\cup\left(\sqrt{\frac{\sqrt{2}-1}{2}},\infty\right)$

10. (d) : Let equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

and $P\left(ae, \frac{b^2}{a}\right)$ be one end of a latus rectum through $S(ae, 0)$.

Now, equation of normal L is $\frac{a^2x}{ae} - \frac{b^2y}{\frac{b^2}{a}} = a^2 - b^2$

$\Rightarrow \frac{ax}{e} - ay = a^2e^2 \Rightarrow \frac{x}{e} - y = ae^2$

This normal is passing through $L'(0, -b)$

$\therefore b = ae^2 \Rightarrow b^2 = a^2e^4$

$\therefore a^2(1 - e^2) = a^2e^4 \Rightarrow e^4 + e^2 - 1 = 0$

11. (b) : Let the first term and common difference be α and β respectively.

$\therefore t_{m+1} = \alpha + m\beta, \ t_{n+1} = \alpha + n\beta, \ t_{r+1} = \alpha + r\beta$

Again, we have t_{m+1}, t_{n+1} and t_{r+1} are in G.P.

$\therefore (\alpha + n\beta)^2 = (\alpha + m\beta)(\alpha + r\beta)$

$\Rightarrow \alpha\beta(2n - m - r) = \beta^2(mr - n^2) \Rightarrow \frac{\alpha}{\beta} = \frac{(mr - n^2)}{2n - m - r}$

$\left[\because m, n, r \text{ are in H.P.} \therefore \frac{2}{n} = \frac{1}{m} + \frac{1}{r} \Rightarrow 2mr = n(m + r)\right]$

$\Rightarrow \frac{\alpha}{\beta} = \frac{2mr - 2n^2}{2(2n - m - r)} = \frac{mn + nr - 2n^2}{2(2n - m - r)}$

$= \frac{n(m + r - 2n)}{-2(m + r - 2n)} = -\frac{n}{2}$

12. (b) : Given, $f''(x) = 6(x - 1)$ ...(i)

Integrating (i) on both sides, we get

$f'(x) = 3(x - 1)^2 + c_1$ ...(ii)

$\Rightarrow 3 = 3 + c_1 \{\because f(x) = y = 3x + 5 \therefore f'(x) = 3 \ \forall x \in R\}$

$\Rightarrow c_1 = 0$

Integrating (ii) on both sides, we get

$f(x) = (x - 1)^3 + c_2$

$\Rightarrow 1 = (2 - 1)^3 + c_2 \Rightarrow c_2 = 0$

$\therefore f(x) = (x - 1)^3$

13. (c) : Given, $f(x) = \begin{cases} \frac{\sin^3 x^2}{x} & \forall x \neq 0 \\ 0 & \forall x = 0 \end{cases}$

Now, $\lim_{x \to 0} \frac{\sin^3 x^2}{x} = \lim_{x \to 0} \frac{\sin x^2}{x^2}(x \sin^2 x^2)$

$= \lim_{x \to 0}\left(\frac{\sin x^2}{x^2}\right) \lim_{x \to 0}(x \sin^2 x^2) = 0$

$\therefore \lim_{x \to 0} \frac{\sin^3 x^2}{x} = 0 = f(0)$

So, $f(x)$ is continuous at $x = 0$

Now $Lf'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{\sin^3 h^2}{h} - 0}{h} = 0$

Similarly, $Rf'(0) = 0$

14. (d) : $f(x) = \int_{x^2}^{x^3} \frac{dt}{\log t}, x > 0$

$\Rightarrow f'(x) = \frac{1}{\log x^3} \cdot \frac{d(x^3)}{dx} - \frac{1}{\log x^2} \cdot \frac{d(x^2)}{dx}$

$= \frac{3x^2}{3 \log x} - \frac{2x}{2 \log x} = \frac{x^2 - x}{\log x} = \frac{x(x - 1)}{\log x}$

Now, $f'(x) = 0$ gives $x(x - 1) = 0 \Rightarrow x = 0, x = 1$

Let us consider $x = 1$, as $x > 0$

Hence, for $x < 1$, $f'(x) = (-\text{ve})(-\text{ve}) = (+\text{ve}) > 0$

and for $x > 1$, $f'(x) = (+\text{ve})(+\text{ve}) = (+\text{ve}) > 0$

$\therefore$ $f'(x)$ does not change its sign in the intermediate neighborhood of $x = 1$

So $x = 1$ is neither the point of maxima or minima.

But $f'(x) > 0 \ \forall \ x < 1$ and $x > 1$

Therefore, $f(x)$ is an increasing function $\forall \ x \in R$

15. (c) : For $y = 4x^2$ and $y = 3x$, the point of intersection of curves are $(0,0)$ and $\left(\frac{3}{4}, \frac{9}{4}\right)$.

Therefore, the area enclosed by the above curves is

$$= \int_0^{3/4} (3x - 4x^2)dx = \left[\frac{3}{2}x^2 - \frac{4}{3}x^3\right]_0^{3/4} = \frac{9}{32} = A_1$$

Again the area of the region enclosed by the curves $y = 4x^2$ and $y = |3x|$

$$= 2\int_0^{3/4} (3x - 4x^2)dx = 2 \times \left[\frac{3}{2}x^2 - \frac{4}{3}x^3\right]_0^{3/4} = 2 \times \frac{9}{32} = A_2$$

Therefore, $A_1 : A_2 = 1 : 2$

16. (c) : The given differential equation is

$$\frac{dt}{dx} = \frac{t\left(\frac{dg(x)}{dx}\right) - t^2}{g(x)}$$

$$\Rightarrow \frac{dt}{dx} = t\frac{g'(x)}{g(x)} - \frac{t^2}{g(x)} \Rightarrow \frac{dt}{dx} - t\frac{g'(x)}{g(x)} = -\frac{t^2}{g(x)}$$

$$\Rightarrow -\frac{1}{t^2}\frac{dt}{dx} + \frac{1}{t}\frac{g'(x)}{g(x)} = \frac{1}{g(x)}$$

$$\Rightarrow \frac{dz}{dx} + z\frac{g'(x)}{g(x)} = \frac{1}{g(x)} \left[\text{Let } \frac{1}{t} = z \therefore -\frac{1}{t^2}\frac{dt}{dx} = \frac{dz}{dx}\right]$$

The above equation is of type $\frac{dy}{dx} + Py = Q$

$$\therefore \ \text{I.F.} = e^{\int \frac{g'(x)}{g(x)}dx} = g(x)$$

So, solution is $zg(x) = \int \frac{1}{g(x)} g(x)dx = x + c$

$$\Rightarrow \frac{1}{t}g(x) = x + c \Rightarrow t = \frac{g(x)}{x + c}$$

17. (a) : Let $z = e^{i\theta}$ $\therefore$ $\arg z = \theta$

Also suppose that $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$

$$\therefore \ z_1Rz_2 \Leftrightarrow |\arg z_1 - \arg z_2| = \frac{2\pi}{3}$$

$$\therefore \ z_1Rz_2 \Leftrightarrow |\theta_1 - \theta_2| = \frac{2\pi}{3} \Leftrightarrow z_2Rz_1$$

but $z_1 \neq z_2$ (if $z_1 = z_2$ then $\frac{2\pi}{3} = 0$ which is not true).

$\Rightarrow$ R is symmetric.

18. (c) : We have, $\lim_{x \to 0} \frac{f(x)}{x} = 2$

$$\Rightarrow \lim_{x \to 0} \frac{\min\left\{\sin\sqrt{[m]}x, |x|\right\}}{x} = 2$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin\sqrt{[m]}x}{x} = 2 \qquad [\because \sin \lambda x \le |x| \ \forall \ \lambda, x]$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin\sqrt{[m]}x}{\sqrt{[m]}x} \times \sqrt{[m]} = 2$$

$$\Rightarrow \sqrt{[m]} = 2 \Rightarrow [m] = 4 \Rightarrow 4 \le m < 5$$

19. (c) : Let θ be the angle between the line and plane.

$$\therefore \ \sin\theta = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}$$

$$[\text{where } \vec{b} = 2\hat{i} - 3\hat{j} + 2\hat{k}, \vec{n} = 3\hat{i} - 2\hat{j} + 5\hat{k}]$$

$$\Rightarrow \ \theta = \sin^{-1}\left(\frac{22}{\sqrt{646}}\right)$$

20. (b) : Given line is $\frac{x-3}{2} = \frac{y-4}{5} = \frac{z-6}{7}$.

Thus direction ratios are 2, 5, 7 (say a_1, b_1, c_1).

Given plane is $2x - 5y + 3z = 9$. Thus the direction ratios to the normal to the plane are 2, –5, 3 (say a_2, b_2, c_2)

Since $a_1a_2 + b_1b_2 + c_1c_2 = 4 - 25 + 21 = 0$, therefore the line is parallel to the plane.

21. (660) : The number of ways in which 2R's are never together $= \frac{7!}{2!2!} - \frac{6!}{2!} = \frac{5040}{4} - \frac{720}{2} = 1260 - 360 = 900$

The number of ways in which 2A's are together but not two 2R's $= 4! \times \frac{^5P_2}{2} = 24 \times 10 = 240$

Now, there are in all 900 arrangements in each of which the two R's never together. In any one such arrangement, either the two A's are together or two A's are not together.

Since number of all such arrangements in which two A's are together = 240, therefore, the number of arrangements in which neither two R's nor 2A's are together = 900 – 240 = 660

22. (485) : The number of ways of selections of 3 man's relative and 3 wife's relative are shown in the following table

Case	Man's relative(7)		Wife's relative(7)	
	3 men	4 women	4 men	3 women
(i)	0	3	3	0
(ii)	1	2	2	1
(iii)	2	1	1	2
(iv)	3	0	0	3

The required number of ways are

$({}^3C_0 \times {}^4C_3 \times {}^4C_3 \times {}^3C_0) + ({}^3C_1 \times {}^4C_2 \times {}^4C_2 \times {}^3C_1)$
$+ ({}^3C_2 \times {}^4C_1 \times {}^4C_1 \times {}^3C_2) + ({}^3C_3 \times {}^4C_0 \times {}^4C_0 \times {}^3C_3)$
$= 16 + 324 + 144 + 1 = 485$

23. (7) : $\log_{\sqrt{7}} x + \log_{\sqrt[3]{7}} x + \log_{\sqrt[4]{7}} x +$ ... upto 20 terms $= 230$

$\Rightarrow \log_7 x\{(1 + 2 + 3 + 4 + ... + 20 + 21) - 1\} = 230$

$\Rightarrow 230(\log_7 x) = 230 \Rightarrow \log_7 x = 1 \Rightarrow x = 7$

24. (254) : Here the given equation is

$p(x^2 - x) + x + 5 = 0 \Rightarrow px^2 - (p - 1)x + 5 = 0$

$\therefore \alpha + \beta = \frac{p-1}{p}$ and $\alpha\beta = \frac{5}{p}$

Now $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5} \Rightarrow \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{4}{5}$

$\Rightarrow \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{4}{5} \Rightarrow \frac{(p-1)^2 - 10p}{5p} = \frac{4}{5}$

$\Rightarrow p^2 - 16p + 1 = 0 \quad \therefore \quad p_1 + p_2 = 16$ and $p_1 p_2 = 1$

Now, $\frac{p_1}{p_2} + \frac{p_2}{p_1} = \frac{(p_1 + p_2)^2 - 2p_1p_2}{p_1p_2} = \frac{256 - 2}{1} = 254$

25. (1) : $\because \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

$\Rightarrow \tan 30° = \frac{3\tan 10° - \tan^3 10°}{1 - 3\tan^2 10°}$ (Putting $A = 10°$)

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{3\tan 10° - \tan^3 10°}{1 - 3\tan^2 10°}$

$\Rightarrow 1 - 3\tan^2 10° = \sqrt{3}(3\tan 10° - \tan^3 10°)$

$\Rightarrow 1 + 9\tan^4 10° - 6\tan^2 10° = 3(9\tan^2 10° + \tan^6 10° - 6\tan^4 10°)$

$\Rightarrow 3\tan^6 10° - 27\tan^4 10° + 33\tan^2 10° = 1$

JEE 2020 PRACTICE PAPER ADVANCED

Exam on 23rd August

One or More Than One Option(s) Correct Type

1. If four distinct points on the curve $y = 2x^4 + 7x^3 + 3x - 5$ are collinear, then the A.M. of the x-coordinate of the four points is

(a) $-\frac{7}{8}$ (b) $\frac{3}{4}$ (c) $\frac{7}{8}$ (d) $-\frac{3}{4}$

2. The differential equation for the family of curves $y = c \sin x$ can be given by

(a) $\left(\frac{dy}{dx}\right)^2 = y^2 \cot^2 x$

(b) $\left(\frac{dy}{dx}\right)^2 - \left(\sec x \frac{dy}{dx}\right)^2 + y^2 = 0$

(c) $\left(\frac{dy}{dx}\right)^2 = \tan^2 x$ (d) $\frac{dy}{dx} = y \cot x$

3. Let $ABCD$ be a tetrahedron. Let 'a' be the length of edge AB and let Δ be the area of projection of the tetrahedron on a plane, perpendicular to AB, then volume of the tetrahedron is

(a) $\frac{a\Delta}{2}$ (b) $\frac{a\Delta}{3}$ (c) $\frac{a\Delta}{4}$ (d) $\frac{a\Delta}{5}$

4. If a, b, c are the sides of a triangle, then

$\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a}$ can take value(s)

(a) 1 (b) 2 (c) 3 (d) 4

5. $I = \int \frac{(1-\cos\theta)^{2/7}}{(1+\cos\theta)^{9/7}} d\theta$ is equal to

(a) $\tan\frac{\theta}{2} + c$ (b) $\frac{7}{11}\left[\tan\left(\frac{\theta}{2}\right)^{11/7}\right] + c$

(c) $7\left(\tan\frac{\theta}{2}\right)^7 + c$ (d) none of these

6. If $b_{n+1} = \frac{1}{1-b_n}$ for $n \geq 1$ and $b_1 = b_3$,

then $\sum_{r=1}^{2001} b_r^{2001}$ is equal to

(a) 2001 (b) –2001
(c) 0 (d) none of these

7. If α, β and γ are the roots of the equation $x^2(px + q) = r(x + 1)$. Then, the value of determinant

$\begin{vmatrix} 1+\alpha & 1 & 1 \\ 1 & 1+\beta & 1 \\ 1 & 1 & 1+\gamma \end{vmatrix}$ is

(a) $\alpha\beta\gamma$ (b) $1+\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

(c) 0 (d) none of these

8. A, B, C are three events for which $P(A) = 0.4$, $P(B) = 0.6$, $P(C) = 0.5$, $P(A \cup B) = 0.75$, $P(A \cap C) = 0.35$ and $P(A \cap B \cap C) = 0.2$. If $P(A \cup B \cup C) \geq 0.75$, then $P(B \cap C)$ can take value(s)

(a) 0.1 (b) 0.2 (c) 0.3 (d) 0.5

9. The value of $\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}$ is

(a) 1 (b) $\frac{\sqrt{7}}{2}$ (c) $\frac{3\sqrt{3}}{4}$ (d) $\frac{\sqrt{15}}{4}$

10. $f(x)$ is defined for $x \geq 0$ and has a continuous derivative. It satisfies $f(0) = 1, f'(0) = 0$ and $(1 + f(x)) f''(x) = 1 + x$. The values that $f(1)$ can't take is (are)

(a) 2 (b) 1.75 (c) 1.50 (d) 1.35

9. A man wants to divide 101 coins, a rupee each, among his 3 sons with the condition that no one receives more money than the combined total of other two. The number of ways of doing this is

(a) ${}^{103}C_2 - 3\ {}^{52}C_2$ (b) $\frac{{}^{103}C_2}{3}$

(c) 1275 (d) $\frac{{}^{103}C_2}{6}$

12. The greatest of the numbers $1, 2^{1/2}, 3^{1/3}, 4^{1/4}, 5^{1/5}, 6^{1/6}$ and $7^{1/7}$ is

(a) $2^{1/2}$ (b) $3^{1/3}$
(c) $7^{1/7}$ (d) all are equal

13. If $f(x)=\begin{cases} x^2+2, & x<0 \\ 3, & x=0 \\ x+2, & x>0 \end{cases}$, then

(a) $f(x)$ has a maximum at $x = 0$
(b) $f(x)$ is strictly decreasing on the left of 0
(c) $f'(x)$ is strictly increasing on the left of 0
(d) $f'(x)$ is strictly increasing on the right of 0

14. Let $\frac{\tan\left(\frac{\pi}{4}+\alpha\right)}{5}=\frac{\tan\left(\frac{\pi}{4}+\beta\right)}{3}=\frac{\tan\left(\frac{\pi}{4}+\gamma\right)}{2}$.

Then, $12\sin^2(\alpha - \beta) + 15\sin^2(\beta - \gamma) - 7\sin^2(\gamma - \alpha)$ is equal to

(a) $-1/2$ (b) $1/2$ (c) 1 (d) 0

15. Let a and b be positive real numbers, then $\log(a^{10}) + {}^{10}C_1\log(a^9b) + {}^{10}C_2\log(a^8b^2) + ... + \log(b^{10}) = \log(ab)^{\lambda}$, where value of λ is

(a) 5120 (b) 2048 (c) 1024 (d) 10240

16. A curve passes through (5, 4) and the slope of tangent at (x, y) is $\frac{1-x}{y-1}$, then

(a) The area of triangle formed by tangent and normal at (5, 4) and the x-axis is 50/3 sq. units
(b) the area of triangle formed by tangent and normal at (5, 4) and the x-axis is 86/3 sq. units
(c) The length of perpendicular from (0, 0) to tangent at (5, 4) is 32/5 units
(d) The length of perpendicular from (0, 0) to tangent at (5, 4) is 16/5 units

17. Radius of circle which is drawn on a normal chord of $y^2 = 4x$ as diameter and it passes through vertex is

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $2\sqrt{3}$ (d) $3\sqrt{3}$

18. Let $f : R \to R$ be a function defined by $f(x+1)=\frac{f(x)-5}{f(x)-3}\forall\in R$. Then which of the following statement(s) is/are true?

(a) $f(2008) = f(2004)$ (b) $f(2006) = f(2010)$
(c) $f(2006) = f(2002)$ (d) $f(2006) = f(2018)$

19. A straight line through a point $P(\alpha, 2)$, $(\alpha \neq 0)$ meets the ellipse $4x^2 + 9y^2 = 36$ at A and D and meets the coordinate axes at B and C, such that PA, PB, PC, PD are in G.P. then a possible value of α can be

(a) 6 (b) 8/3 (c) 2 (d) 4

20. If E and F are two independent events, such that $P(E\cap F)=\frac{1}{6}$, $P(E^C\cap F^C)=\frac{1}{3}$ and $(P(E) - P(F))(1 - P(F)) > 0$, then

(a) $P(E)=\frac{1}{2}$ (b) $P(E)=\frac{1}{4}$
(c) $P(F)=\frac{1}{3}$ (d) $P(F)=\frac{2}{3}$

Comprehension Type

Paragraph for Q. No. 21 to 23

Let n be a positive integer such that $I_n=\int x^n\sqrt{a^2-x^2}\,dx$.

21. The value of I_1 is

(a) $\frac{2}{3}(a^2-x^2)^{1/2}+C$ (b) $\frac{1}{3}(a^2-x^2)^{3/2}+C$
(c) $-\frac{2}{3}(a^2-x^2)^{3/2}+C$ (d) $-\frac{1}{3}(a^2-x^2)^{3/2}+C$

22. The value of the expression $\frac{\int_0^a x^4\sqrt{a^2-x^2}\,dx}{\int_0^a x^2\sqrt{a^2-x^2}\,dx}$ is

(a) $\frac{a^2}{6}$ (b) $\frac{3a^2}{2}$ (c) $\frac{3a^2}{4}$ (d) $\frac{a^2}{2}$

23. If $I_n=\frac{-x^{n-1}(a^2-x^2)^{3/2}}{n+2}+kI_{n-2}$, then the values k is

(a) $\frac{n-1}{n+2}$ (b) $\frac{n+2}{n-1}$
(c) $\left(\frac{n-1}{n+2}\right)a^2$ (d) $\left(\frac{n+2}{n-1}\right)a^2$

Paragraph for Q. No. 24 to 26

In a problem of differentiation of $\frac{f(x)}{g(x)}$, one student writes the derivative of $\frac{f(x)}{g(x)}$ as $\frac{f'(x)}{g'(x)}$ and he finds the correct result if $g(x) = x^2$ and $\lim_{x\to\infty} f(x)=4$. A circle 'C' of minimum radius is drawn which intersects both the curves $y = f(x)$ and $y = g(x)$ at two points at which they intersect. Let 'P' be a point on $y = g(x)$.

24. $\lim_{x\to\infty}\left(\frac{f(x)}{4}\right)^{\frac{x^2}{2x+1}}$

(a) e (b) 1 (c) e^{-2} (d) e^2

25. $\sum_{r=1}^{n+2} \frac{r^2}{f(r)}$ is

(a) $\frac{n(n+1)(2n+1)}{24}$ (b) $\frac{n(n+1)(2n+1)}{6}$

(c) $\frac{1}{4}+\frac{n(n+1)(2n+1)}{24}$ (d) $1+\frac{n(n+1)(2n+1)}{24}$

26. Coordinates of 'P' at which tangent to $y = g(x)$ is parallel to common chord of $y = f(x)$ and $y = g(x)$ are

(a) (1/2, 1/4) (b) (2, 4)

(c) (4, 16) (d) (0,0)

Matrix-Match Type

27. Match the following .

Column-I		Column-II	
A.	Normal of parabola $y^2 = 4x$ at P and Q meets at $R(x_2, 0)$ and tangents at P and Q meets at $T(x_1, 0)$, then if $x_2 = 3$, then the area of quadrilateral $PTQR$ is	(i)	3
B.	The length of latus rectum plus tangent PT will be	(ii)	6
C.	The quadrilateral $PTQR$ can be inscribed in a circle, then the value of $\frac{\text{circumference}}{4\pi}$ will be	(iii)	1
D.	The number of normals that can be drawn to the parabola from $R(x_2, 0)$	(iv)	8

28. Match the following :

Column-I		Column-II	
A.	Number of solutions of the equation $\sin^{-1}x+\cos^{-1}x^2=\frac{\pi}{2}$ is	(i)	1
B.	The number of ordered pairs (x, y) satisfying $\frac{\sin^{-1}x}{x}+\frac{\sin^{-1}y}{y}=2$ is	(ii)	2
C.	Number of solutions of the equation $\cos(\cos x) = \lvert\sin(\sin x)\rvert$ is	(iii)	0
D.	Number of solutions of the equation $\tan\left(x+\frac{\pi}{6}\right)=2\tan x$ is	(iv)	3

Numerical Value Type

29. The number of real solutions of the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ is ______.

30. A, B and C respectively take turns tossing a die. A begins, then B and then C and then again A. The probability that C will be the first one to toss a 6 is $\frac{m}{n}$, where m and n are in reduced form, then sum of digits of m is ______.

31. The number of solutions that the equation $\sin(\cos(\sin x)) = \cos(\sin(\cos x))$ has in $\left[0, \frac{\pi}{2}\right]$ is

32. The minimum distance between the curves $y=\sqrt{2-x^2}$, $y=\frac{9}{x}$; $x, y>0$ is k, then k^2 equals ______.

33. Let $S=\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+.....+\sqrt{1+\frac{1}{1999^2}+\frac{1}{2000^2}}$, then find $|2000(S - 2000)|$.

34. If $f: [-1, 1] \to R$ be a continuous function satisfying $f(2x^2 - 1) = (x^3 + x) f(x)$, then find $\lim_{x\to 0} \frac{f(\cos x)}{\sin x}$.

35. A sequence is obtained by deleting all perfect squares from set of natural numbers. The remainder when the 2003^{rd} term of new sequence is divided by 2048, is ______.

36. If the equation of the curve on reflection of the ellipse $\frac{(x-4)^2}{16}+\frac{(y-3)^2}{9}=1$ about the line $x-y-2=0$ is $16x^2 + 9y^2 + k_1x - 36y + k_2 = 0$, then $\frac{k_1+k_2}{44}$ is ______.

Solutions

1. **(a) :** Let all the four points of the curve lie on the line $y = mx + c$

$\therefore\ 2x^4 + 7x^3 + 3x - 5 = mx + c$

$\Rightarrow\ 2x^4 + 7x^3 + (3 - m)x - 5 - c = 0$

Let it has four roots a, b, c, d.

$$\text{Sum of root} = \frac{\text{Coefficient of } x^3}{\text{Coefficent of } x^4}$$

$\Rightarrow\ a + b + c + d = -\frac{7}{2}$

$\therefore \quad A.M. = \dfrac{a+b+c+d}{4} = -\dfrac{7}{8}$

2. **(a, b, d) :** We have, $y = c\sin x$ ..(i)

$$\Rightarrow \quad \frac{dy}{dx} = c\cos x \qquad \text{...(ii)}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = c^2\cos^2 x \qquad \text{...(iii)}$$

Putting $c = \dfrac{y}{\sin x}$ from (i) in (iii), we get

$$\left(\frac{dy}{dx}\right)^2 = y^2\cot^2 x$$

Similarly, putting $c = \dfrac{y}{\sin x}$ from (i) in (ii), we get

$$\frac{dy}{dx} = y\cot x$$

Also (iii) can be written as

$$\left(\frac{dy}{dx}\right)^2 = c^2(1-\sin^2 x) = c^2 - c^2\sin^2 x = c^2 - y^2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\right)^2 \sec^2 x - y^2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 - \left(\sec x\frac{dy}{dx}\right)^2 + y^2 = 0$$

3. **(b) :** Let $A(0, 0, 0)$, $B(0, 0, a)$, $C(p, b, c)$, $D(\alpha, \beta, \gamma)$. Let projection of C on XY plane is $C' \equiv (p, b, 0)$ and projection of D on XY plane is $D' \equiv (\alpha, \beta, 0)$.

$$\text{Area of } \Delta AD'C' = \frac{1}{2}\left|\overrightarrow{AD'} \times \overrightarrow{AC'}\right| = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha & \beta & 0 \\ p & b & 0 \end{vmatrix}$$

$\Rightarrow \quad |\alpha b - \beta p| = 2\Delta$ ... (i)

So, volume of tetrahedron $ABCD$,

$$V = \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]$$

$$= \frac{1}{6}\begin{vmatrix} 0 & 0 & a \\ p & b & c \\ \alpha & \beta & \gamma \end{vmatrix} = \frac{a}{6} \times 2\Delta \qquad \text{[Using (i)]}$$

Hence, $V = \dfrac{a\Delta}{3}$

4. **(c, d) :** $c+a-b$, $b+c-a$, $a+b-c$ are all positive.

$$\therefore \quad \frac{\dfrac{a}{c+a-b} + \dfrac{b}{a+b-c} + \dfrac{c}{b+c-a}}{3} \geq \left[\frac{abc}{(c+a-b)(a+b-c)(b+c-a)}\right]^{1/3} \qquad \text{...(i)}$$

Also, $a^2 \geq a^2 - (b-c)^2 \Rightarrow a^2 \geq (a+b-c)(a-b+c)$

Similarly, $b^2 \geq (b+c-a)(b-c+a)$,

$c^2 \geq (c+a-b)(c-a+b)$

$\therefore \quad a^2b^2c^2 \geq (a+b-c)^2(b+c-a)^2(c+a-b)^2$

Thus $abc \geq (a+b-c)(b+c-a)(c+a-b)$

$$\Rightarrow \quad \frac{abc}{(c+a-b)(a+b-c)(b+c-a)} \geq 1$$

So, from (i) $\dfrac{a}{c+a-b} + \dfrac{b}{a+b-c} + \dfrac{c}{b+c-a} \geq 3$

5. **(b) :** $I = \displaystyle\int \frac{(1-\cos\theta)^{2/7}}{(1+\cos\theta)^{9/7}}\, d\theta = \frac{1}{2}\int \frac{\left(\sin\frac{\theta}{2}\right)^{4/7}}{\left(\cos\frac{\theta}{2}\right)^{18/7}}\, d\theta$

Put $\dfrac{\theta}{2} = t \Rightarrow d\theta = 2dt$

$$\therefore \quad I = \int \frac{(\sin t)^{4/7}}{(\cos t)^{18/7}}\, dt = \int (\tan t)^{4/7}\sec^2 t\, dt$$

Put, $\tan t = u \Rightarrow \sec^2 t\, dt = du$

$$\therefore \quad I = \int u^{4/7}\, du = \frac{7}{11}\left[\tan\left(\frac{\theta}{2}\right)\right]^{11/7} + c$$

6. **(b) :** Given, $b_{n+1} = \dfrac{1}{1-b_n}$

$$b_2 = \frac{1}{1-b_1},\ b_3 = \frac{1}{1-b_2} = \frac{1}{1-\frac{1}{1-b_1}} = \frac{1-b_1}{-b_1} = \frac{b_1-1}{b_1}$$

As $b_1 = b_3 \Rightarrow b_1^2 - b_1 + 1 = 0$

$$\Rightarrow \quad b_1 = -\omega \text{ or } -\omega^2 \Rightarrow b_2 = \frac{1}{1+\omega} = -\omega \text{ or } -\omega^2$$

$$\therefore \quad \sum_{r=1}^{2001} b_r^{2001} = 2001(-1)^{2001} = -2001$$

7. **(c) :** Given equation, $px^3 + qx^2 - rx - r = 0$ having roots α, β, γ. So, we have

$$\alpha+\beta+\gamma = \frac{-q}{p};\ \alpha\beta+\beta\gamma+\gamma\alpha = -\frac{r}{p};\ \alpha\beta\gamma = \frac{r}{p}$$

$$\therefore \quad \Delta = \alpha\beta\gamma\left(1+\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right) = \alpha\beta\gamma\left(\frac{\alpha\beta\gamma+\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}\right) = 0$$

8. **(a, b, c) :** $P(A\cup B\cup C) = P(A)+P(B)+P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$

$= 0.4 + 0.6 + 0.5 + 0.75 - (0.4 + 0.6) - P(B\cap C) - 0.35 + 0.2$

[since $P(A\cup B) = P(A) + P(B) - P(A\cap B)$]

$= 1.1 - P(B\cap C)$

But, $0.75 \leq P(A\cup B\cup C) \leq 1$

$\Rightarrow \quad 0.75 \leq 1.1 - P(B\cap C) \leq 1 \Rightarrow 0.1 \leq P(B\cap C) \leq 0.35$

9. **(b) :** Let $S=\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}$ and

$C=\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}$

Now, $C + iS = \alpha + \alpha^2 + \alpha^4$ ...(i)

where, $\alpha=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$, is complex 7^{th} root of unity

Also, $C - iS = \alpha^6 + \alpha^5 + \alpha^3$ ...(ii)

$\therefore\ \alpha^6=\bar{\alpha}, \alpha^5=\bar{\alpha}^2, \alpha^3=\bar{\alpha}^4$

Adding (i) and (ii), we get

$2C=\alpha+\alpha^2+\alpha^3+...+\alpha^6=\frac{\alpha^7-\alpha}{\alpha-1}=-1 \qquad (\because \alpha^7=1)$

$\Rightarrow\ C=-1/2$

Multiplying (i) and (ii), we get

$C^2+S^2=2 \Rightarrow S=\frac{\sqrt{7}}{2}$

10. **(a, b, c, d) :** $1 + x$ is never zero, so $1 + f(x)$ is never zero. It is 1 for $x = 0$, so it is always positive.

Hence $f''(x)$ is always positive.

$f'(0) = 0$, so $f'(x) > 0$ for all $x > 0$ and hence f is strictly increasing.

So, in particular, $1 + f(x) \geq 2$ for all x.

We have $f''(x)\leq\frac{(1+x)}{2}$

Integrating, $f'(x)\leq f'(0)+\frac{x}{2}+\frac{x^2}{4}$

Integrating again, $f(x)\leq f(0)+\frac{x^2}{4}+\frac{x^3}{12}$,

Hence $f(1)\leq 1+\frac{1}{4}+\frac{1}{12}=\frac{4}{3}$

11. **(a) :** Let the amount received by the sons ₹ x, ₹ y and ₹ z respectively, then

$x \leq y + z = 101 - x$

i.e., $2x \leq 101$ and $x + y + z = 101$

$\therefore\ x \leq 50, y \leq 50, z \leq 50$

The corresponding multinomial is $(1 + x + x^2 +...+ x^{50})^3$

$=\left(\frac{x^{51}-1}{x-1}\right)^3=-\left(x^{51}-1\right)^3(1-x)^{-3}$

$=-(x^{153} - 1 - 3x^{102} + 3x^{51})(1 + {}^3C_1x + {}^4C_2x^2 + {}^5C_3x^3 + ...+ {}^{52}C_{50}\,x^{50} + ...+ {}^{103}C_{101}x^{101}$

$\therefore$ Required number of (distribution) ways

= coefficient of x^{101} in the expansion of $(1 + x + x^2 + ...+ x^{50})^3$

$= 1 \times {}^{103}C_{101} - 3({}^{52}C_{50}) = {}^{103}C_2 - 3\ {}^{52}C_2$

12. **(b) :** Assuming the function

$f(x) = x^{1/x}\ \{x > 0\}$

$\therefore\ f'(x)=x^{1/x}\left[\frac{d}{dx}\left(\frac{1}{x}\log x\right)\right]=x^{1/x}\left[-\frac{\log x}{x^2}+\frac{1}{x^2}\right]$

$\Rightarrow\ f'(x)=\frac{x^{1/x}}{x^2}(1-\log x)$

Clearly, when $x > e$, $f'(x)$ will be negative and $f(x)$ is decreasing.

When $0 < x < e$, $f'(x)$ will be positive and $f(x)$ is increasing.

It means $3^{1/3} > 4^{1/4} > 5^{1/5} > 6^{1/6} > 7^{1/7}$

Now compare (1) and $(2)^{1/2}$ and $(3)^{1/3}$

Apply same power on given three numbers.

$(1)^6$		$(2)^{1/2 \times 6}$		$(3)^{1/3 \times 6}$
1		2^3		3^2
1	<	8	<	9
(1)	<	$(2)^{1/2}$	<	$(3)^{1/3}$

So, maximum number is $(3)^{1/3}$.

13. **(a, b, c) :** $f(x)=\begin{cases}x^2+2, & x<0\\ 3, & x=0\\ x+2, & x>0\end{cases}$

$f(0)=3,\ \lim_{x\to 0^-} f(x)=2,\ \lim_{x\to 0^+} f(x)=2$

$\therefore\ f(x)$ has a maximum at $x = 0$

$f'(x) = 2x, x < 0$

$\therefore\ f'(x) < 0$ for $x < 0$

$\therefore\ f(x)$ is strictly decreasing on the left of 0

$f''(x) = 2, x < 0$

$\therefore\ f''(x) > 0, x < 0$

$\therefore\ f'(x)$ is strictly increasing on the left of 0.

14. **(d) :** $\frac{\tan\left(\frac{\pi}{4}+\alpha\right)}{\tan\left(\frac{\pi}{4}+\beta\right)}=\frac{5}{3}$

Applying componendo-dividendo rule, we get

$\frac{\sin\left(\frac{\pi}{4}+\alpha+\frac{\pi}{4}+\beta\right)}{\sin\left(\frac{\pi}{4}+\alpha-\frac{\pi}{4}-\beta\right)}=\frac{5+3}{5-3} \Rightarrow \frac{\cos(\alpha+\beta)}{\sin(\alpha-\beta)}=4$

$\Rightarrow\ 4\sin(\alpha - \beta) = \cos(\alpha + \beta)$ .. (i)

Now, $\frac{\tan\left(\frac{\pi}{4}+\beta\right)}{\tan\left(\frac{\pi}{4}+\gamma\right)}=\frac{3}{2}$

$$\Rightarrow \frac{\sin\left(\frac{\pi}{4}+\beta+\frac{\pi}{4}+\gamma\right)}{\sin\left(\frac{\pi}{4}+\beta-\frac{\pi}{4}-\gamma\right)}=\frac{3+2}{3-2} \Rightarrow \frac{\cos(\beta+\gamma)}{\sin(\beta-\gamma)}=5$$

$$\Rightarrow 5\sin(\beta-\gamma)=\cos(\beta+\gamma) \quad \text{... (ii)}$$

$$\text{and } \frac{\tan\left(\frac{\pi}{4}+\gamma\right)}{\tan\left(\frac{\pi}{4}+\alpha\right)}=\frac{2}{5} \Rightarrow \frac{\sin\left(\frac{\pi}{4}+\gamma+\frac{\pi}{4}+\alpha\right)}{\sin\left(\frac{\pi}{4}+\gamma-\frac{\pi}{4}-\alpha\right)}=\frac{2+5}{2-5}$$

$$\Rightarrow \frac{\cos(\gamma+\alpha)}{\sin(\gamma-\alpha)}=\frac{7}{-3}$$

$$\Rightarrow -\frac{7}{3}\sin(\gamma-\alpha)=\cos(\gamma+\alpha) \quad \text{... (iii)}$$

Now multiplying (i) by 3 sin $(\alpha-\beta)$, (ii) by 3 $\sin(\beta-\gamma)$, (iii) by 3 sin $(\gamma-\alpha)$ and adding, we get $12\sin^2(\alpha-\beta)+15\sin^2(\beta-\gamma)-7\sin^2(\gamma-\alpha)$

$$=\frac{3}{2}\{(\sin 2\alpha-\sin 2\beta)+(\sin 2\beta-\sin 2\gamma)+(\sin 2\gamma-\sin 2\alpha)\}=0$$

15. (a) : We have,

$$\log(a^{10})+{}^{10}C_1\log(a^9b)+{}^{10}C_2\log(a^8b^2)+...+\log(b^{10})=\log(ab)^\lambda$$

$$\Rightarrow \log(a^{10})+\log(b^{10})+{}^{10}C_1[\log(a^9b)+\log(ab^9)]+{}^{10}C_2[\log(a^8b^2)+\log(a^2b^8)]+{}^{10}C_3[\log(a^7b^3)+\log(a^3b^7)+{}^{10}C_4[\log(a^6b^4)+\log(a^4b^6)]+{}^{10}C_5a^5b^5=\log(ab)^\lambda$$

$$\Rightarrow \log(ab)^{10}+{}^{10}C_1\log(ab)^{10}+...+{}^{10}C_4\log(ab)^{10}+\frac{{}^{10}C_5\log(ab)^{10}}{2}=\log(ab)^\lambda$$

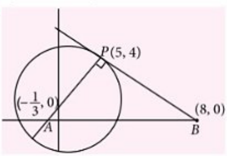
$$\Rightarrow \log(ab)^{10}\left[1+{}^{10}C_1+{}^{10}C_2+{}^{10}C_3+{}^{10}C_4+\frac{{}^{10}C_5}{2}\right]=\log(ab)^\lambda$$

$$\Rightarrow \log(ab)^{10}[1+10+45+120+210+126]=\log(ab)^\lambda$$

$$\Rightarrow (512)\log(ab)^{10}=\log(ab)^\lambda$$

$$\Rightarrow 5120\log(ab)=\lambda\log(ab) \therefore \lambda=5120.$$

16. (a, c) :



Given, $\frac{dy}{dx}=\frac{1-x}{y-1} \Rightarrow \frac{(y-1)^2}{2}=\frac{-(x-1)^2}{2}+C$

$\because$ It passes through (5, 4) $\therefore (x-1)^2+(y-1)^2=25$

$\Rightarrow x^2+y^2-2x-2y-23=0$

Equation of tangent at (5, 4) is

$5x+4y-(x+5)-(y+4)-23=0$

$\Rightarrow 4x+3y-32=0$ ...(i)

and equation of normal at (5, 4) is $y-4=\frac{4-1}{5-1}(x-5)$

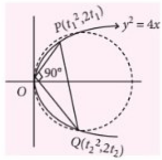
$\Rightarrow y-4=\frac{3}{4}(x-5)$

$\Rightarrow 3x-4y+1=0$ ... (ii)

$\therefore$ Area of $\Delta ABP=\frac{1}{2}\times 4\times\left(8+\frac{1}{3}\right)=\frac{50}{3}$ sq. units

$\therefore$ Length of perpendicular from (0, 0) to (i) is $\left|\frac{0+0-32}{5}\right|=\frac{32}{5}$ units

17. (d) : Normal chord is PQ and normal is drawn at $P(at_1^2, 2at_1)$



$\therefore t_2=-t_1-\frac{2}{t_1}$ ...(i)

Since, $OP \perp OQ$

$$\Rightarrow \left(\frac{2t_1-0}{t_1^2-0}\right)\cdot\left(\frac{2t_2-0}{t_2^2-0}\right)=-1$$

$$\Rightarrow \left(\frac{2}{t_1}\cdot\frac{2}{t_2}\right)=-1$$

$\therefore t_1t_2=-4$ ...(ii)

From (i)and (ii), $-\frac{4}{t_1}=-t_1-\frac{2}{t_1}$

$$\Rightarrow \frac{2}{t_1}=t_1 \Rightarrow t_1=\sqrt{2}, t_2=-\sqrt{2}-\frac{2}{\sqrt{2}}$$

$$P\equiv(2, 2\sqrt{2}),\ Q\equiv(8,-4\sqrt{2})$$

$$\therefore PQ=\sqrt{36+72}=\sqrt{108}$$

Hence, $r=\frac{PQ}{2}=\sqrt{27}=3\sqrt{3}$

18. (a, b, c, d) : $f(x+1)=\frac{f(x)-5}{f(x)-3}$ ....(i)

$$\Rightarrow f(x)\cdot f(x+1)-3f(x+1)=f(x)-5$$

$$\Rightarrow f(x)=\frac{3f(x+1)-5}{f(x+1)-1}$$

Replacing x by $(x-1)$, we have

$$f(x-1)=\frac{3f(x)-5}{f(x)-1} \quad \text{...(ii)}$$

Using (i), we have

$$f(x+2)=\frac{f(x+1)-5}{f(x+1)-3}=\frac{\frac{f(x)-5}{f(x)-3}-5}{\frac{f(x)-5}{f(x)-3}-3}=\frac{2f(x)-5}{f(x)-2} \quad \text{...(iii)}$$

Using (ii), we have

$$f(x-2)=\frac{3f(x-1)-5}{f(x-1)-1}=\frac{3\left(\frac{3f(x)-5}{f(x)-1}\right)-5}{\frac{3f(x)-5}{f(x)-1}-1}=\frac{2f(x)-5}{f(x)-2} \quad \text{...(iv)}$$

From (iii) and (iv), we have

$f(x+2)=f(x-2) \Rightarrow f(x+4)=f(x)$

$\Rightarrow$ $f(x)$ is periodic with period 4.

19. (a) : Any point on the line through $P(\alpha, 2)$ is $(\alpha + r\cos\theta, 2 + r\sin\theta)$

So, $\frac{(\alpha+r\cos\theta)^2}{9}+\frac{(2+r\sin\theta)^2}{4}=1$

gives $PA\cdot PD=\frac{4\alpha^2}{4\cos^2\theta+9\sin^2\theta}$

Similarly, $PB\cdot PC=\frac{2\alpha}{\sin\theta\cos\theta}$

Hence, $\frac{4\alpha^2}{4\cos^2\theta+9\sin^2\theta}=\frac{2\alpha}{\sin\theta\cos\theta}$

i.e., $2\alpha\sin 2\theta+5\cos 2\theta=13 \Rightarrow |\alpha|\geq 6$.

20. (a, c) : $P(E\cap F)=P(E)\cdot P(F)=\frac{1}{6}$ ... (i)

$P(E^c\cap F^c)=(1-P(E))(1-P(F))=\frac{1}{3}$

$\Rightarrow P(E)+P(F)=\frac{5}{6}$ ... (ii)

$\Rightarrow |P(E)-P(F)|=\frac{1}{6}$

As $(P(E)-P(F))(1-P(F))>0$

$\Rightarrow P(E)>P(F) \Rightarrow P(E)-P(F)=\frac{1}{6}$ ... (iii)

Solving (ii) and (iii), we get $P(E)=\frac{1}{2},\ P(F)=\frac{1}{3}$

21. (d) : $I_1=\int x\sqrt{a^2-x^2}\,dx$,

Put $a^2-x^2=t \Rightarrow xdx=-\frac{1}{2}dt$

$\therefore I_1=-\frac{1}{2}\int\sqrt{t}\,dt=-\frac{1}{3}t^{3/2}=-\frac{1}{3}(a^2-x^2)^{3/2}+C$

22. (d) : $\int_0^a x^4\sqrt{a^2-x^2}\,dx$

$$=\left[\frac{-x^3(a^2-x^2)^{3/2}}{3}\right]_0^a+\int_0^a x^2(a^2-x^2)^{3/2}dx$$

$$=\int_0^a(a^2-x^2)x^2\sqrt{a^2-x^2}\,dx$$

$$=\int_0^a a^2\left(x^2\sqrt{a^2-x^2}\right)dx-\int_0^a x^4\sqrt{a^2-x^2}\,dx$$

$$=\frac{a^2}{2}\int_0^a x^2\sqrt{a^2-x^2}\,dx \quad \therefore \quad \frac{\int_0^a x^4\sqrt{a^2-x^2}\,dx}{\int_0^a x^2\sqrt{a^2-x^2}\,dx}=\frac{a^2}{2}$$

23. (c) : $I_n=\int x^n\sqrt{a^2-x^2}\,dx=\int x^{n-1}(x\sqrt{a^2-x^2})\,dx$

$$=x^{n-1}\left[-\frac{1}{3}(a^2-x^2)^{3/2}\right]+\frac{n-1}{3}\int x^{n-2}(a^2-x^2)\sqrt{a^2-x^2}\,dx$$

$$=-\frac{1}{3}x^{n-1}(a^2-x^2)^{3/2}+\frac{n-1}{3}a^2I_{n-2}-\frac{n-1}{3}I_n$$

$$\Rightarrow\left(1+\frac{n-1}{3}\right)I_n=-\frac{1}{3}x^{n-1}(a^2-x^2)^{3/2}+\frac{a^2(n-1)}{3}I_{n-2}$$

$$\Rightarrow I_n=\frac{-x^{n-1}}{n+2}(a^2-x^2)^{3/2}+\left(\frac{n-1}{n+2}\right)a^2I_{n-2}$$

$$\therefore\quad k=\left(\frac{n-1}{n+2}\right)a^2$$

24. (d) : Consider

$$\left(\frac{f(x)}{x^2}\right)'=\frac{x^2f'(x)-f(x)2x}{x^4}=\frac{xf'(x)-2f(x)}{x^3}$$

By wrong calculations,

$$\left(\frac{f(x)}{x^2}\right)'=\frac{f'(x)}{2x} \quad \therefore \quad \frac{f'(x)}{2x}=\frac{xf'(x)-2f(x)}{x^3}$$

$\Rightarrow x^2f'(x)=2xf'(x)-4f(x)$

$\Rightarrow 4f(x)=(2x-x^2)f'(x)$

$$\Rightarrow \frac{4}{2x-x^2}=\frac{f'(x}{f(x)} \quad \therefore \quad \frac{f'(x)}{f(x)}=2\left(\frac{1}{x}+\frac{1}{2-x}\right)$$

On integrating, we get

$\therefore \ln f(x)=2[\ln x-\ln(2-x)]+\ln c$

$$\Rightarrow f(x)=c\cdot\frac{x^2}{(2-x)^2} \quad \therefore \lim_{x\to\infty}f(x)=c=4$$

$$\therefore f(x)=\frac{4x^2}{(2-x)^2}$$

Now, $\lim_{x\to\infty}\left(\frac{f(x)}{4}\right)^{\frac{x^2}{2x+1}}=\lim_{x\to\infty}\left(\frac{x^2}{(x-2)^2}\right)^{\frac{x^2}{2x+1}}=e^2$

25. (c) : $\sum_{r=1}^{n+2}\frac{r^2}{f(r)}=\sum_{r=1}^{n+2}\frac{r^2(r-2)^2}{4r^2}=\frac{1}{4}\sum_{r=1}^{n+2}(r-2)^2$

$=\frac{1}{4}\left(1+\sum_{r=3}^{n+2}(r-2)^2\right)=\frac{1}{4}\left(1+\frac{n(n+1)(2n+1)}{6}\right)$

26. (b) : Points of intersection of $y=x^2$ and $y=\frac{4x^2}{(x-2)^2}$ are given by $\frac{x^2}{1}=\frac{4x^2}{(x-2)^2}$

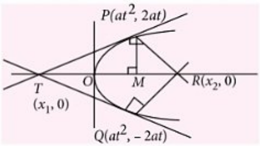
$\Rightarrow\ x^2-4x=0\ \Rightarrow\ x=0, x=4$

$\therefore$ Points of intersection are (0, 0) and (4, 16)

Slope of the common chord = 4

$g'(x)=2x=4\Rightarrow x=2\ \therefore$ Required point is (2, 4).

27. (A) - (iv) ; (B) - (ii) ; (C) - (iii) ; (D) - (i)



(A) $y^2=4x$ ...(i)

Clearly, if $P(at^2, 2at)$ then by symmetry, $Q(at^2, -2at)$

Equation of tangent is $ty=x+at^2$

For t; $y=0$, $x_1=-at^2$

and equation of normal is $y=-tx+2at+at^3$

For R; $y=0$, $x_2=(2a+at^2)$ ...(ii)

Here, $a=1\Rightarrow x_1=-t^2$ and $x_2=(2+t^2)$

$\Rightarrow\ 3=2+t^2\Rightarrow t^2=1\ \Rightarrow\ t=\pm1$

Take $t=1$, then $x_1=-1\ \therefore PM=2at=2\times1\times1=2$

$RT=x_1+x_2=(1+3)=4$

$\therefore$ Area of quadrilateral $PTQR=2\times\left(\frac{1}{2}\times4\times2\right)$

= 8 sq. units

(B) Length of latus rectum + tangent PT

$=4a+\sqrt{S_1}=4\times1+\sqrt{0^2-4(1)}=4+2=6$ units

(C) Clearly, RT will be the diameter of circle

$\therefore$ Circumference $=(\pi\times\text{diameter})=\pi\times RT=\pi\times4$

$\therefore\ \frac{\text{Circumference}}{4\pi}=\frac{\pi\times4}{4\pi}=1$

(D) Since in the part (1), we have found $x_2=(2a+at^2)>2a$, (if $t\neq0$)

$\therefore$ For three real normals, $x_2>2a=2\times1=2$

i.e. $x_2=2$ of straight lines otherwise a rectangular hyperbola.

28. (A) → (ii); (B) → (iii); (C) → (iii); (D) → (iii)

(A) $\sin^{-1}x+\cos^{-1}x^2=\frac{\pi}{2}$

$\Rightarrow\ \cos^{-1}x^2=\cos^{-1}x\Rightarrow x^2=x\Rightarrow x=0, 1$

(B) We have, $\frac{\sin^{-1}x}{x}+\frac{\sin^{-1}y}{y}=2$

$\frac{\sin^{-1}x}{x}$ is increasing $x\geq0$ and decreasing for $x\leq0$

$\Rightarrow\ \frac{\sin^{-1}x}{x}>1$ and $\frac{\sin^{-1}y}{y}>1$

$\therefore\ \frac{\sin^{-1}x}{x}+\frac{\sin^{-1}y}{y}=2$ has no solution.

(C) We have, $\cos(\cos x)=|\sin(\sin x)|$

$\Rightarrow\ \cos x=2n\pi\pm\frac{\pi}{2}\pm\sin x$

$\Rightarrow\cos x\pm\sin x=2n\pi\pm\frac{\pi}{2}\ \therefore$ no solution exist

(D) $\tan\left(x+\frac{\pi}{6}\right)=2\tan x$ let $\tan x=y$

$\Rightarrow\ 2y^2-\sqrt{3}y+1=0$.

Since, $D<0\ \therefore$ No real solution exist.

29. (0) : Given, $e^{\sin x}-e^{-\sin x}-4=0$

Let $e^{\sin x}=y$ $[y>0\ \forall\ x\in R]$

Then, $y-1/y-4=0\Rightarrow\ y^2-4y-1=0$

$\Rightarrow\ y=\frac{4\pm\sqrt{16+4}}{2}=2\pm\sqrt{5}$

y can never be negative. So, $2-\sqrt{5}$ can not be accepted.

Now, $2+\sqrt{5}>e$ and maximum value of $e^{\sin x}=e$.

Hence, $e^{\sin x}\neq2+\sqrt{5}$ *i.e.*, there is no solution.

30. (7): Let P be the probability that C wins then,

$P=\left(\frac{5}{6}\right)^2\times\frac{1}{6}+\left(\frac{5}{6}\right)^5\times\frac{1}{6}+\left(\frac{5}{6}\right)^8\times\frac{1}{6}+....$

$=\left(\frac{5}{6}\right)^2\times\frac{1}{6}\left[1+\left(\frac{5}{6}\right)^3+\left(\frac{5}{6}\right)^6+...\right]$

$=\left(\frac{5}{6}\right)^2\times\frac{1}{6}\left[\frac{1}{1-\left(\frac{5}{6}\right)^3}\right]=\left(\frac{5}{6}\right)^2\times\frac{1}{6}\times\frac{216}{91}=\frac{25}{91}$

$\therefore\ m=2+5=7$

31. (1) : Let $f(x)=\sin(\cos(\sin x))-\cos(\sin(\cos x))$

$\Rightarrow\ f'(x)=-\cos(\cos(\sin x))\sin(\sin x)(\cos x)$

$-\sin(\sin(\cos x))\cos(\cos x)\sin x$

$\Rightarrow f'(x) < 0 \forall x \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow f(x)$ is decreasing in $\left[0, \frac{\pi}{2}\right]$

and $f(0) = \sin 1 - \cos(\sin 1)$ ...(i)

Now, $\sin 1 - \cos(\sin 1) = \cos\left(\frac{\pi}{2} - 1\right) - \cos(\sin 1)$,

$\sin 1 > \sin\frac{\pi}{4} > \frac{1}{\sqrt{2}} \therefore \frac{\pi}{2} - 1 < \sin 1 \Rightarrow \sin 1 > \frac{\pi - 2}{2}$

$\therefore$ $\sin 1 > \cos(\sin 1)$ from (i)

$\therefore f(0) = \sin 1 - \cos(\sin 1) > 0,\ f\left(\frac{\pi}{2}\right) = \sin(\cos 1) - 1 < 0$

$\therefore$ There will be only one root lies between $\left[0, \frac{\pi}{2}\right]$.

32. (8) : Equation of normal at $\left(\alpha, \frac{9}{\alpha}\right)$ to $xy = 9$ is

$$y - \frac{9}{\alpha} = \frac{\alpha^2}{9}(x - \alpha) \quad \text{... (i)}$$

To have a shortest distance it must pass through $(0, 0)$. $\therefore \alpha = 3$

$\therefore$ Point on hyperbola is $(3, 3)$.

Similarly, point on circle is $(1, 1)$.

$\therefore S.D. = \sqrt{(3-1)^2 + (3-1)^2} = \sqrt{8} = k$

$\therefore k^2 = 8$

33. (1) : $t_r = \sqrt{1 + \frac{1}{r^2} + \frac{1}{(r+1)^2}} = \sqrt{\frac{r^2 + (r+1)^2 + r^2(r+1)^2}{r^2(r+1)^2}}$

$= \sqrt{\frac{2r^2 + 2r + 1 + r^2(r^2 + 2r + 1)}{r^2(r+1)^2}}$

$= \sqrt{\frac{r^4 + 2r^3 + 3r^2 + 2r + 1}{r^2(r+1)^2}} = \frac{r^2 + r + 1}{r(r+1)} = \frac{1}{r(r+1)} + 1$

$= 1 + \frac{1}{r} - \frac{1}{r+1}$

$\therefore S_n = \sum_{n=1}^{1999}\left(1 + \frac{1}{r} - \frac{1}{r+1}\right) \Rightarrow S = 2000 - \frac{1}{2000}$

Hence, $|2000(S - 2000)| = 1$

34. (0) : $f(2x^2 - 1) = (x^3 + x) f(x)$ ... (i)

Replacing x by $-x$

$f(2x^2 - 1) = -(x^3 + x) f(-x)$ ... (ii)

From (i) and (ii), we get $f(-x) = -f(x)$

Hence, $f(x)$ is an odd function and as it is continuous

$\Rightarrow f(0) = 0$

Now, $\lim_{x \to 0} \frac{f(2x^2 - 1)}{x} = \lim_{x \to 0} (x^2 + 1) f(x)$

$\lim_{x \to 0} f(x) = 0$ $(\because f(x)$ is continuous at $x = 0)$

$\Rightarrow \lim_{x \to 0} \frac{f(2x^2 - 1)}{x} = 0$

Let $x = \sin\frac{\theta}{2}$

$\therefore \lim_{\theta \to 0} \frac{f\left(2\sin^2\frac{\theta}{2} - 1\right)}{\sin\frac{\theta}{2}} = 0 \Rightarrow \lim_{\theta \to 0} \frac{f(\cos\theta)}{\sin\frac{\theta}{2}} = 0$ ... (iii)

$\Rightarrow \lim_{x \to 0} \frac{f(\cos x)}{2\sin\frac{x}{2}\cos\frac{x}{2}} = 0 \Rightarrow \lim_{x \to 0} \frac{f(\cos x)}{\sin x} = 0$

35. (0) : Since, $\left[\sqrt{2046}\right] = \left[\sqrt{2047}\right] = \left[\sqrt{2048}\right]$

$= \left[\sqrt{2049}\right] = 45$

$\therefore$ 2003^{rd} term is $2003 + 45 = 2048$

Hence, remainder is 0

36. (3) : Let $P(4(1 + \cos\theta), 3(1 + \sin\theta))$ be any point on the ellipse $\frac{(x-4)^2}{16} + \frac{(y-3)^2}{9} = 1$

Let the reflection of the point P about the line $x - y - 2 = 0$ be h, k.

Then, $\frac{h - 4(1 + \cos\theta)}{1} = \frac{k - 3(1 + \sin\theta)}{-1}$

$= \frac{-2(4(1 + \cos\theta) - 3(1 + \sin\theta) - 2)}{2} = -(4\cos\theta - 3\sin\theta) + 1$

$\therefore h = 4(1 + \cos\theta) - 4\cos\theta + 3\sin\theta + 1 = 4 + 3\sin\theta + 1 = 3\sin\theta$.

and $k = 3(1 + \sin\theta) + 4\cos\theta - 3\sin\theta - 1 = 3 + 4\cos\theta - 1 = 4\cos\theta$.

Now, $\left(\frac{h-5}{3}\right)^2 + \left(\frac{k-2}{4}\right)^2 = 1$

$\therefore$ Equation of image is $16(x - 5)^2 + 9(y - 2)^2 = 144$

$\Rightarrow 16x^2 + 9y^2 - 160x - 36y + 292 = 0$

$\therefore k_1 = -160, k_2 = 292 \Rightarrow \frac{k_1 + k_2}{44} = \frac{132}{44} = 3$

Monthly Test Drive-1 CLASS XI ANSWER KEY

1. (a)	**2.** (b)	**3.** (c)	**4.** (a)
5. (c)	**6.** (d)	**7.** (b, c)	**8.** (b, d)
9. (b)	**10.** (b, c, d)	**11.** (d)	**12.** (a, c, d)
13. (a, b)	**14.** (c)	**15.** (a)	**16.** (c)
17. (13)	**18.** (8)	**19.** (210)	**20.** (56)

CONCEPT BOOSTERS

Class XI

Circles

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR

A circle is defined as the locus of a point which moves in a plane such that its distance from a fixed point in that plane always remains the same *i.e.*, constant.

The fixed point is called the centre of the circle and the fixed distance is called the radius of the circle.

STANDARD FORMS OF EQUATION OF A CIRCLE

- **General equation of a circle :** The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ where g, f, c are constant.
  - Centre of the circle is $(-g, -f)$.
  - Radius of the circle is $\sqrt{g^2 + f^2 - c}$.

Nature of the circle

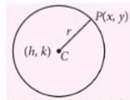
- If $g^2 + f^2 - c > 0$, then the radius of the circle will be real. Hence, in this case, it is possible to draw a circle on a plane.
- If $g^2 + f^2 - c = 0$, then the radius of the circle will be zero. Such a circle is known as point circle.
- If $g^2 + f^2 - c < 0$, then the radius of the circle will be an imaginary number. Hence, in this case, it is not possible to draw a circle.

The condition for the second degree equation to represent a circle : The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle iff

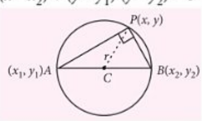
- $a = b \neq 0$
- $h = 0$
- $\Delta = abc + 2hgf - af^2 - bg^2 - ch^2 \neq 0$
- $g^2 + f^2 - ac \geq 0$

- **Central form of equation of a circle :** The equation of a circle having centre (h, k) and radius r is
  $(x - h)^2 + (y - k)^2 = r^2$

  If the centre is origin, then the equation of the circle is $x^2 + y^2 = r^2$.



- **Equation of a circle on a given diameter :** The equation of the circle drawn on the straight line joining two given points (x_1, y_1) and (x_2, y_2) as diameter is
  $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$



EQUATION OF A CIRCLE IN SOME SPECIAL CASES

- If centre of the circle is (h, k) and it passes through origin then its equation is
  $(x - h)^2 + (y - k)^2 = h^2 + k^2 \Rightarrow x^2 + y^2 - 2hx - 2ky = 0$.
- If the circle touches x-axis then its equation is $(x \pm h)^2 + (y \pm k)^2 = k^2$.
- If the circle touches y-axis then its equation is $(x \pm h)^2 + (y \pm k)^2 = h^2$.
- If the circle touches both the axes then its equation is $(x \pm r)^2 + (y \pm r)^2 = r^2$.
- If the circle touches x-axis at origin then its equation is $x^2 + (y \pm k)^2 = k^2 \Rightarrow x^2 + y^2 \pm 2ky = 0$.

*Alok Kumar, a B.Tech from IIT Kanpur and INMO 4th ranker of his time, has been training IIT and Olympiad aspirants for close to two decades now. His students have bagged AIR 1 in IIT JEE and also won medals for the country at IMO. He has also taught at Maths Olympiad programme at Cornell University, USA and UT, Dallas. He has been regularly proposing problems in international Mathematics journals.

- If the circle touches y-axis at origin, the equation of circle is $(x \pm h)^2 + y^2 = h^2 \Rightarrow x^2 + y^2 \pm 2xh = 0$.
- If the circle passes through origin and cuts intercepts a and b on axes, the equation of circle is $x^2 + y^2 - ax - by = 0$ and centre is $C(a/2, b/2)$.

INTERCEPTS ON THE AXES

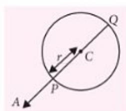
The lengths of intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on x and y axes are $2\sqrt{g^2 - c}$ and $2\sqrt{f^2 - c}$ respectively. Therefore,

- The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the x-axis in real and distinct points, touches or does not meet in real points according as $g^2 >, =$ or $< c$.
- Similarly, the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the y-axis in real and distinct points, touches or does not meet in real points according as $f^2 >, =$ or $< c$.

POSITION OF A POINT WITH RESPECT TO A CIRCLE

A point (x_1, y_1) lies outside, on or inside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ according as $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ is positive, zero or negative.

The least and greatest distance of a point from a circle: Let $S = 0$ be a circle and $A(x_1, y_1)$ be a point. If the diameter of the circle through A is passing through the circle at P and Q, then



(i) the least distance of A from the circle $= AP = |AC - r|$

(ii) the greatest distance of A from the circle $= AQ = |AC + r|$

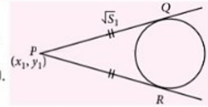
where 'r' is the radius and C is the centre of the circle.

TANGENT TO A CIRCLE AT A GIVEN POINT

- **Point form**
  - The equation of tangent at (x_1, y_1) to circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.
  - The equation of tangent at (x_1, y_1) to circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.
- **Parametric form :** Since parametric co-ordinates of a point on the circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$ then equation of tangent at $(a\cos\theta, a\sin\theta)$ is $x \cdot a\cos\theta + y \cdot a\sin\theta = a^2$ or $x\cos\theta + y\sin\theta = a$.
- **Slope form :** The straight line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$ and the point of contact of tangent $y = mx \pm a\sqrt{1+m^2}$ is $\left(\frac{\mp ma}{\sqrt{1+m^2}}, \frac{\pm a}{\sqrt{1+m^2}}\right)$.

LENGTH OF TANGENT

Let PQ and PR be two tangents drawn from $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Then $PQ = PR$ is called the length of tangent drawn from point P and is given by



$$PQ = PR = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}.$$

PAIR OF TANGENTS

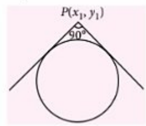
From a given point $P(x_1, y_1)$ two tangents PQ and PR can be drawn to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$

Their combined equation is $SS_1 = T^2$, where $S = 0$ is the equation of circle, $T = 0$ is the equation of tangent at (x_1, y_1) and S_1 is obtained by replacing x by x_1 and y by y_1 in S.

DIRECTOR CIRCLE

The locus of the point of intersection of two perpendicular tangents to a circle is called the director circle.

Let the circle be $x^2 + y^2 = a^2$, then equation of director circle is $x^2 + y^2 = 2a^2$.



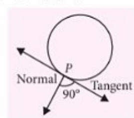
Obviously director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the given circle.

Director circle of $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$.

NORMAL TO A CIRCLE AT A GIVEN POINT

The normal of a circle at any point is a straight line, which is perpendicular to the tangent at the point and always passes through the centre of the circle.

- **Equation of normal:** The equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) is
$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1) \text{ or } \frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}.$$
The equation of normal to the circle $x^2 + y^2 = a^2$ at any point (x_1, y_1) is
$$xy_1 - x_1y = 0 \text{ or } \frac{x}{x_1} = \frac{y}{y_1}.$$

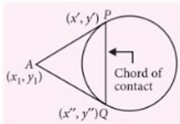


- **Parametric form :** Since parametric co-ordinates of a point on the circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$.
$\therefore$ Equation of normal at $(a\cos\theta, a\sin\theta)$ is
$$\frac{x}{a\cos\theta} = \frac{y}{a\sin\theta} \text{ or } \frac{x}{\cos\theta} = \frac{y}{\sin\theta}$$

or $y = x\tan\theta$ or $y = mx$ where $m = \tan\theta$, which is slope form of normal.

CHORD OF CONTACT OF TANGENTS

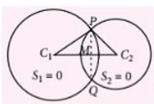
- **Chord of contact :** The chord joining the points of contact of the two tangents to a conic drawn from a given point, outside it, is called the chord of contact of tangents.



- **Equation of chord of contact :** The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.
  Equation of chord of contact at (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.
  It is clear from above that the equation to the chord of contact coincides with the equation of the tangent, if point (x_1, y_1) lies on the circle.
- **Equation of the chord bisected at a given point :** The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ bisected at the point (x_1, y_1) is given by $T = S_1$.
  i.e., $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
  $= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$

COMMON CHORD OF TWO CIRCLES

- **Definition :** The chord joining the points of intersection of two given circles is called their common chord.
- **Equation of common chord :** The equation of the common chord of two circles

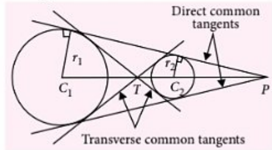


  $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$
  and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$
  is $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$
  i.e., $S_1 - S_2 = 0$
- **Length of the common chord :**
  $PQ = 2(PM) = 2\sqrt{C_1P^2 - C_1M^2}$
  Where C_1P = radius of the circle $S_1 = 0$ and C_1M = length of the perpendicular from the centre C_1 to the common chord PQ.

COMMON TANGENTS TO TWO CIRCLES

Different cases of intersection of two circles :
Let the two circles be $(x - x_1)^2 + (y - y_1)^2 = r_1^2$ .....(i)
and $(x - x_2)^2 + (y - y_2)^2 = r_2^2$ .....(ii)
with centres $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$ and radii r_1 and r_2 respectively. Then following cases may arise :

Case I : When $|C_1C_2| > r_1 + r_2$ *i.e.*, the distance between the centres is greater than the sum of radii.



In this case four common tangents can be drawn to the two circles, in which two are direct common tangents and the other two are transverse common tangents.
The points P, T of intersection of direct common tangents and transverse common tangents respectively, always lie on the line joining the centres of the two circles and divide it externally and internally respectively in the ratio of their radii.

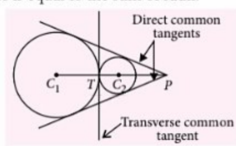
$$\frac{C_1P}{C_2P} = \frac{r_1}{r_2} \text{ (externally) and } \frac{C_1T}{C_2T} = \frac{r_1}{r_2} \text{ (internally)}$$

Hence, the ordinates of P and T are

$$P \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$$

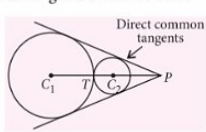
and $$T \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$$

Case II : When $|C_1C_2| = r_1 + r_2$ *i.e.*, the distance between the centres is equal to the sum of radii.

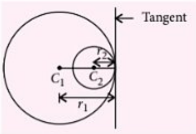


In this case two direct common tangents are real and distinct while the transverse tangents are coincident.

Case III : When $|C_1C_2| < r_1 + r_2$ *i.e.*, the distance between the centres is less than sum of radii. In this case two direct common tangents are real and distinct while the transverse tangents are imaginary.

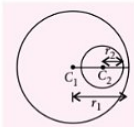


Case IV : When $|C_1C_2| = |r_1 - r_2|$, *i.e.*, the distance between the centres is equal to the difference of the radii. In this case two tangents are real and coincident while the other two tangents are imaginary.



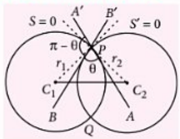
Case V : When $|C_1C_2| < |r_1 - r_2|$, *i.e.*, the distance between the centres is less than the difference of the radii.

In this case, all the four common tangents are imaginary.



ANGLE OF INTERSECTION OF TWO CIRCLES

The angle of intersection between two circles $S = 0$ and $S' = 0$ is defined as the angle between their tangents at their point of intersection.



If $\quad S \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

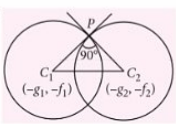
$\quad S' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

are two circles with radii r_1, r_2 and d be the distance between their centres then the angle of intersection θ between them is given by $\cos\theta = \dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$

$$\text{or} \quad \cos\theta = \frac{2(g_1g_2 + f_1f_2) - (c_1 + c_2)}{2\sqrt{g_1^2 + f_1^2 - c_1}\,\sqrt{g_2^2 + f_2^2 - c_2}}.$$

Condition of Orthogonality :

If the angle of intersection of the two circles is a right angle ($\theta = 90°$) , then such circles are called orthogonal circles and condition for orthogonality is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.



FAMILY OF CIRCLES

- The equation of the family of circles passing through the point of intersection of two given circles $S = 0$ and $S' = 0$ is given as $S + \lambda S' = 0$, (where λ is a parameter, $\lambda \neq -1$)
- The equation of the family of circles passing through the point of intersection of circle $S = 0$ and a line $L = 0$ is given as $S + \lambda L = 0$, (where λ is a parameter)
- The equation of the family of circles touching the circle $S = 0$ and the line $L = 0$ at their point of contact P is $S + \lambda L = 0$, (where λ is a parameter)
- The equation of a family of circles passing through two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ can be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(where λ is a parameter)

- The equation of family of circles, which touch $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any finite m is $(x - x_1)^2 + (y - y_1)^2 + \lambda\{(y - y_1) - m(x - x_1)\} = 0$
  And if m is infinite, the family of circles is $(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1) = 0$, (where λ is a parameter)

RADICAL AXIS

The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.

The equation of the radical axis of the two circles $S_1 = 0$, $S_2 = 0$ is $S_1 - S_2 = 0$ *i.e.*, $2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$, which is a straight line.

Properties of the radical axis

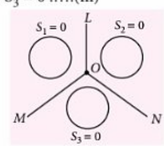
- The radical axis and common chord are identical for two intersecting circles.
- The radical axis is perpendicular to the straight line which joins the centres of the circles.
- If two circles cut a third circle orthogonally, the radical axis of the two circles will pass through the centre of the third circle.

RADICAL CENTRE

The radical axes of three circles, taken in pairs, meet at a point, which is called their radical centre. Let the three circles be

$S_1 = 0$ .....(i), $S_2 = 0$ .....(ii) and $S_3 = 0$ .....(iii)

Let the straight lines *i.e.*, OL and OM meet at O. The equation of any straight line passing through O is $(S_1 - S_2) + \lambda(S_3 - S_1) = 0$, where λ is any constant.



For $\lambda = 1$, this equation become $S_2 - S_3 = 0$, which is, equation of ON.

Thus the third radical axis also passes through the point where the straight lines OL and OM meet.

In the above figure O is the radical centre.

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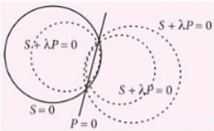
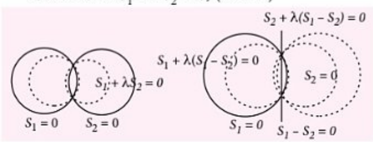
Co-axial system of circles

A system (family) of circles, every pair of which have the same radical axis, are called co-axial circles.



- The equation of a system of co-axial circles, when the equation of the radical axis and of one circle of the system are $P \equiv lx + my + n = 0$, $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ respectively, is $S + \lambda P = 0$ (λ is an arbitrary constant).
- The equation of a co-axial system of circles, where the equation of any two circles of the system are
$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$
and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ respectively, is $S_1 + \lambda(S_1 - S_2) = 0$ or $S_2 + \lambda_1(S_1 - S_2) = 0$
Other form $S_1 + \lambda S_2 = 0$, $(\lambda \neq -1)$



- The equation of a system of co-axial circles in the simplest form is $x^2 + y^2 + 2gx + c = 0$, where g is a variable and c is a constant.

LIMITING POINTS

Limiting points of a system of co-axial circles are the centres of the point circles belonging to the family (Circles whose radii are zero are called point circles).

Let the circle is $x^2 + y^2 + 2gx + c = 0$ .....(i)

where g is a variable and c is a constant.

$\therefore$ Centre and radius of (i) are $(-g, 0)$ and $\sqrt{g^2 - c}$ respectively. Let $\sqrt{g^2 - c} = 0 \Rightarrow g = \pm\sqrt{c}$

Thus we get the two limiting points of the given co-axial system as $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$.

Clearly the above limiting points are real and distinct, real and coincident or imaginary according as $c >, =, < 0$.

IMPORTANT RESULTS

- If two tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are perpendicular to each other, then $g^2 + f^2 = 2c$.
- If the tangent to the circle $x^2 + y^2 = r^2$ at the point (a, b) meets the coordinate axes at the points A and B and O is the origin, then the area of the triangle OAB is $\frac{r^4}{2ab}$.
- The angle between the tangents from (α, β) to the circle $x^2 + y^2 = a^2$ is $2\tan^{-1}\left(\frac{a}{\sqrt{\alpha^2 + \beta^2 - a^2}}\right)$.
- If OA and OB are the tangents from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and C is the centre of the circle, then the area of the quadrilateral $OACB$ is $\sqrt{c(g^2 + f^2 - c)}$.
- If the circle $x^2 + y^2 + 2gx + c^2 = 0$ and $x^2 + y^2 + 2fy + c^2 = 0$ touch each other, then $\frac{1}{g^2} + \frac{1}{f^2} = \frac{1}{c^2}$.
- If the line $lx + my + n = 0$ is a tangent to the circle $(x - h)^2 + (y - k)^2 = a^2$, then $(hl + km + n)^2 = a^2(l^2 + m^2)$.
- If O is the origin and OP, OQ are tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then the circum-centre of the triangle OPQ is $\left(\frac{-g}{2}, \frac{-f}{2}\right)$.
- The length of the chord intercept by the circle $x^2 + y^2 = r^2$ on the line $\frac{x}{a} + \frac{y}{b} = 1$ is
$$2\sqrt{\left(\frac{r^2(a^2 + b^2) - a^2b^2}{a^2 + b^2}\right)}$$
- The length of the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y - b)^2 = b^2$ is $\frac{2ab}{\sqrt{a^2 + b^2}}$.
- The distance between the chord of contact of the tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f) is $\frac{1}{2}\frac{g^2 + f^2 - c}{\sqrt{g^2 + f^2}}$.
- Locus of mid point of a chords of a circle $x^2 + y^2 = a^2$, which subtends an angle α at the centre is $x^2 + y^2 = (a\cos\alpha/2)^2$.
- The locus of mid point of chords of circle $x^2 + y^2 = a^2$, which are making right angle at centre is $x^2 + y^2 = a^2/2$.

PROBLEMS

Single Correct Answer Type

1. The equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ externally at the point $(5, 5)$, is

(a) $x^2 + y^2 - 18x - 16y - 120 = 0$

(b) $x^2 + y^2 - 18x - 16y + 120 = 0$

(c) $x^2 + y^2 + 18x + 16y - 120 = 0$
(d) $x^2 + y^2 + 18x - 16y + 120 = 0$

2. The equation of the circle which touches x-axis and whose centre is $(1, 2)$, is
(a) $x^2 + y^2 - 2x + 4y + 1 = 0$
(b) $x^2 + y^2 - 2x - 4y + 1 = 0$
(c) $x^2 + y^2 + 2x + 4y + 1 = 0$
(d) $x^2 + y^2 + 4x + 2y + 1 = 0$

3. The locus of the centre of the circle which cuts off intercepts of length $2a$ and $2b$ from x-axis and y-axis respectively, is
(a) $x + y = a + b$ (b) $x^2 + y^2 = a^2 + b^2$
(c) $x^2 - y^2 = a^2 - b^2$ (d) $x^2 + y^2 = a^2 - b^2$

4. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then the radius of the circle is
(a) 3/2 (b) 3/4 (c) 1/10 (d) 1/20

5. If the vertices of a triangle be $(2, -2)$, $(-1, -1)$ and $(5, 2)$, then the equation of its circumcircle is
(a) $x^2 + y^2 + 3x + 3y + 8 = 0$
(b) $x^2 + y^2 - 3x - 3y - 8 = 0$
(c) $x^2 + y^2 - 3x + 3y + 8 = 0$
(d) None of these

6. The equation of a circle which touches both axes and the line $3x - 4y + 8 = 0$ and whose centre lies in the third quadrant is
(a) $x^2 + y^2 - 4x + 4y - 4 = 0$
(b) $x^2 + y^2 - 4x + 4y + 4 = 0$
(c) $x^2 + y^2 + 4x + 4y + 4 = 0$
(d) $x^2 + y^2 - 4x - 4y - 4 = 0$

7. The equation of the circle having centre $(1, -2)$ and passing through the point of intersection of lines $3x + y = 14$ and $2x + 5y = 18$ is
(a) $x^2 + y^2 - 2x + 4y - 20 = 0$
(b) $x^2 + y^2 - 2x - 4y - 20 = 0$
(c) $x^2 + y^2 + 2x - 4y - 20 = 0$
(d) $x^2 + y^2 + 2x + 4y - 20 = 0$

8. Equation of the circle which touches the lines $x = 0$, $y = 0$ and $3x + 4y = 4$ is
(a) $x^2 - 4x + y^2 + 4y + 4 = 0$
(b) $x^2 - 4x + y^2 - 4y + 4 = 0$
(c) $x^2 + 4x + y^2 + 4y + 4 = 0$
(d) $x^2 + 4x + y^2 - 4y + 4 = 0$

9. A circle touches x-axis and cuts off a chord of length $2l$ from y-axis. The locus of the centre of the circle is
(a) a straight line (b) a circle
(c) an ellipse (d) a hyperbola

10. The equation of the circle which passes through the points $(2, 3)$ and $(4, 5)$ and the centre lies on the straight line $y - 4x + 3 = 0$, is
(a) $x^2 + y^2 + 4x - 10y + 25 = 0$
(b) $x^2 + y^2 - 4x - 10y + 25 = 0$
(c) $x^2 + y^2 - 4x - 10y + 16 = 0$
(d) $x^2 + y^2 - 14y + 8 = 0$

11. A circle is concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and has area double of its area. The equation of the circle is
(a) $x^2 + y^2 - 6x + 12y - 15 = 0$
(b) $x^2 + y^2 - 6x + 12y + 15 = 0$
(c) $x^2 + y^2 - 6x + 12y + 45 = 0$
(d) None of these

12. The equation of the circle passing through the point $(2, 1)$ and touching y-axis at the origin is
(a) $x^2 + y^2 - 5x = 0$ (b) $2x^2 + 2y^2 - 5x = 0$
(c) $x^2 + y^2 + 5x = 0$ (d) None of these

13. The number of circles touching the lines $x = 0$, $y = a$ and $y = b$ is
(a) One (b) Two (c) Four (d) Infinite

14. The locus of a point which moves such that the sum of the squares of its distances from the three vertices of a triangle is constant, is a circle whose centre is at the
(a) incentre of the triangle
(b) centroid of the triangle
(c) orthocentre of the triangle
(d) None of these

15. The equation of a circle passing through the point $(4, 5)$ and having the centre at $(2, 2)$ is
(a) $x^2 + y^2 + 4x + 4y - 5 = 0$
(b) $x^2 + y^2 - 4x - 4y - 5 = 0$
(c) $x^2 + y^2 - 4x = 13$
(d) $x^2 + y^2 - 4x - 4y + 5 = 0$

16. The locus of the centre of a circle of radius 2 which rolls on the outside of circle $x^2 + y^2 + 3x - 6y - 9 = 0$, is
(a) $x^2 + y^2 + 3x - 6y + 5 = 0$
(b) $x^2 + y^2 + 3x - 6y - 31 = 0$
(c) $x^2 + y^2 + 3x - 6y + \frac{29}{4} = 0$
(d) None of these

17. Area of the circle in which a chord of length $\sqrt{2}$ makes an angle $\frac{\pi}{2}$ at the centre (in sq. units) is

(a) $\frac{\pi}{2}$ (b) 2π (c) π (d) $\frac{\pi}{4}$

18. A line is drawn through a fixed point $P(\alpha, \beta)$ to cut the circle $x^2 + y^2 = r^2$ at A and B. Then $PA \cdot PB$ is equal to

(a) $(\alpha + \beta)^2 - r^2$ (b) $\alpha^2 + \beta^2 - r^2$
(c) $(\alpha - \beta)^2 + r^2$ (d) None of these

19. The equation of the circumcircle of the triangle formed by the lines $x = 0, y = 0, 2x + 3y = 5$ is

(a) $x^2 + y^2 + 2x + 3y - 5 = 0$
(b) $6(x^2 + y^2) - 5(3x + 2y) = 0$
(c) $x^2 + y^2 - 2x - 3y + 5 = 0$
(d) $6(x^2 + y^2) + 5(3x + 2y) = 0$

20. The equation of the circle whose diameter lies on $2x + 3y = 3$ and $16x - y = 4$ which passes through (4,6) is

(a) $5(x^2 + y^2) - 3x - 8y = 200$
(b) $x^2 + y^2 - 4x - 8y = 200$
(c) $5(x^2 + y^2) - 4x = 200$
(d) $x^2 + y^2 = 40$

21. The equation of the circle passing through the point $(-2, 4)$ and through the points of intersection of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ and the line $3x + 2y - 5 = 0$, is

(a) $x^2 + y^2 + 2x - 4y - 4 = 0$
(b) $x^2 + y^2 + 4x - 2y - 4 = 0$
(c) $x^2 + y^2 - 3x - 4y = 0$
(d) $x^2 + y^2 - 4x - 2y = 0$

22. The centre of the circle $x = 2 + 3\cos\theta, y = 3\sin\theta - 1$ is

(a) (3, 3) (b) (2, –1) (c) (–2, 1) (d) (–1, 2)

23. Four distinct points $(2k, 3k)$, (1, 0), (0, 1) and (0, 0) lie on a circle for

(a) $\forall k \in I$ (b) $k < 0$
(c) $0 < k < 1$ (d) For two values of k

24. If one end of the diameter is (1, 1) and other end lies on the line $x + y = 3$, then locus of centre of circle is

(a) $x + y = 1$ (b) $2(x - y) = 5$
(c) $2x + 2y = 5$ (d) None of these

25. The line $x\cos\alpha + y\sin\alpha = p$ will be a tangent to the circle $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = 0$, if $p =$

(a) 0 or a (b) $3a$ (c) $2a$ (d) 0 or $2a$

26. The equations of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are

(a) $x = 0, y = 0$
(b) $(h^2 - r^2)x - 2rhy = 0, x = 0$
(c) $y = 0, x = 4$
(d) $(h^2 - r^2)x + 2rhy = 0, x = 0$

27. If the line $lx + my = 1$ be a tangent to the circle $x^2 + y^2 = a^2$, then the locus of the point (l, m) is

(a) a straight line (b) a circle
(c) a parabola (d) an ellipse

28. The equations of the tangents to the circle $x^2 + y^2 = a^2$ parallel to the line $\sqrt{3}x + y + 3 = 0$ are

(a) $\sqrt{3}x + y \pm 2a = 0$ (b) $\sqrt{3}x + y \pm a = 0$
(c) $\sqrt{3}x + y \pm 4a = 0$ (d) None of these

29. The area of the triangle formed by the tangents from the points (h, k) to the circle $x^2 + y^2 = a^2$ and the line joining their points of contact is

(a) $a\frac{(h^2+k^2-a^2)^{3/2}}{h^2+k^2}$ (b) $a\frac{(h^2+k^2-a^2)^{1/2}}{h^2+k^2}$

(c) $\frac{(h^2+k^2-a^2)^{3/2}}{h^2+k^2}$ (d) $\frac{(h^2+k^2-a^2)^{1/2}}{h^2+k^2}$

30. The equations of the normals to the circle $x^2 + y^2 - 8x - 2y + 12 = 0$ at the points whose ordinate is –1, will be

(a) $2x - y - 7 = 0, 2x + y - 9 = 0$
(b) $2x + y + 7 = 0, 2x + y + 9 = 0$
(c) $2x + y - 7 = 0, 2x + y + 9 = 0$
(d) $2x - y + 7 = 0, 2x - y + 9 = 0$

31. Two tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ will be perpendicular to each other, if

(a) $g^2 + f^2 = 2c$ (b) $g = f = c^2$
(c) $g + f = c$ (d) None of these

32. A tangent to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$_____ the circle $x^2 + y^2 - 8x + 6y + 20 = 0$

(a) touches (b) cuts at real points
(c) cuts at imaginary points
(d) None of these

33. The line $y = mx + c$ intersects the circle $x^2 + y^2 = r^2$ at two real distinct points, if

(a) $-r\sqrt{1+m^2} < c \le 0$ (b) $0 \le c < r\sqrt{1+m^2}$
(c) both (a) and (b) (d) $-c\sqrt{1-m^2} < r$

Multiple Correct Answer Type

34. If $16l^2 + 9m^2 = 24lm + 6l + 8m + 1$ and S be the equation of circle having $lx + my + 1 = 0$ is tangent then

(a) equation of director circle of S is $x^2 + y^2 - 6x - 8y - 25 = 0$
(b) radius of circle is 5
(c) perpendicular distance from centre of S to $x - y + 1 = 0$ is $\sqrt{2}$
(d) equation of circle S is $x^2 + y^2 + 6x + 8y = 0$

35. A point $P(x, y)$ is called a lattice point if $x, y \in I$ (set of integers). Then the total number of lattice points in the interior of the circle $x^2 + y^2 = a^2, a \neq 0$ cannot be

(a) 1996 (b) 1998 (c) 1999 (d) 2001

36. If the line $3x - 4y - \lambda = 0$ touches the circle $x^2 + y^2 - 4x - 8y - 5 = 0$ at (a, b), then $\lambda + a + b$ is equal to

(a) 20 (b) 22 (c) −30 (d) −28

37. In a variable ΔABC, the base BC is fixed and $\angle BAC = \alpha$ (a constant). Then,

(a) the locus of centroid of ΔABC lies on a circle.
(b) the locus of incentre of ΔABC lies on a circle.
(c) the locus of orthocentre of ΔABC lies on a circle.
(d) the locus of excentre opposite to 'A' lies on a circle.

38. A circle is inscribed in a trapezium in which one of the non-parallel sides is perpendicular to the two parallel sides. Then

(a) the diameter of the inscribed circle is the geometric mean of the lengths of the parallel sides.
(b) the diameter of the inscribed circle is the harmonic mean of the lengths of the parallel sides.
(c) the area of the trapezium is the area of the rectangle having lengths of its sides as the lengths of the parallel sides of the trapezium.
(d) the area of the trapezium is half the area of the rectangle having lengths of its sides as the lengths of the parallel sides of the trapezium.

39. Two circles $x^2 + y^2 + px + py - 7 = 0$ and $x^2 + y^2 - 10x + 2py + 1 = 0$ will cut orthogonally if the value of p is

(a) −2 (b) −3 (c) 2 (d) 3

40. The equation of a circle is $S_1 \equiv x^2 + y^2 = 1$. The orthogonal tangents to S_1 meet at another circle S_2 and the orthogonal tangents to S_2 meet at the third circle S_3. Then

(a) Radius of S_2 and S_3 are in the ratio $1 : \sqrt{2}$.
(b) Radius of S_2 and S_3 are in the ratio 1 : 2.
(c) The circles S_1, S_2 and S_3 are concentric.
(d) None of these

Comprehension Type

Paragraph for Q. No. 41 to 43

A system of circles is said to be coaxal when every pair of the circles has the same radical axis. It follows from this definition that

1. The centres of all circles of a coaxal system lie on one straight line, which is perpendicular to the common radical axis.

2. Circles passing through two fixed points form a coaxal system with line joining the points as common radical axis.

3. The equation to a coaxal system of which two members are $S_1 = 0$ and $S_2 = 0$ is $S_1 + \lambda S_2 = 0$, λ is parameter. If we choose the line of centres as x-axis and the common radical axis as y-axis, then the simplest form of equation of coaxal circles is

$$x^2 + y^2 + 2gx + c = 0 \quad \text{...(i)}$$

where c is fixed and g is variable.

If $g = \pm\sqrt{c}, c > 0$, then the radius $g^2 - c$ vanishes and the circles become point circles. The points $(\pm\sqrt{c}, 0)$ are called the limiting points of the system of coaxal circles given by (i).

41. The coordinates of the limiting points of the coaxal system to which the circles $x^2 + y^2 + 4x + 2y + 5 = 0$ and $x^2 + y^2 + 2x + 4y + 7 = 0$ belong are

(a) (0, −3), (0, 3) (b) (0, 3), (−2, −1)
(c) (−2, −1), (0, −3) (d) (2, 1), (−2, −1)

42. The equation to the circle which belongs to the coaxal system of which the limiting points are (1, −1), (2, 0) and which passes through the origin is

(a) $x^2 + y^2 - 4x = 0$ (b) $x^2 + y^2 + 4x = 0$
(c) $x^2 + y^2 - 4y = 0$ (d) $x^2 + y^2 + 4y = 0$

43. If origin be a limiting point of a coaxal system one of whose member is $x^2 + y^2 - 2\alpha x - 2\beta y + c = 0$, then the other limiting point is

(a) $\left(\frac{c\alpha}{\alpha^2+\beta^2}, -\frac{c\beta}{\alpha^2+\beta^2}\right)$ (b) $\left(\frac{c\alpha}{\alpha^2+\beta^2}, \frac{c\beta}{\alpha^2+\beta^2}\right)$

(c) $\left(\frac{\alpha\beta}{\alpha^2+\beta^2}, \frac{c\alpha}{\alpha^2+\beta^2}\right)$ (d) $\left(-\frac{c\beta}{\alpha^2+\beta^2}, \frac{c\alpha}{\alpha^2+\beta^2}\right)$

Paragraph for Q. No. 44 to 46

Let $ABCD$ is a rectangle with $AB = a$ and $BC = b$. A circle is drawn passing through A and B and touching side CD. Another circle is drawn passing through B and C and touching side AD. Let r_1 and r_2 be the radii of these two circles respectively.

44. r_1 equals

(a) $\frac{4b^2 - a^2}{8b}$ (b) $\frac{4b^2 + a^2}{8b}$

(c) $\frac{4a^2 + b^2}{8a}$ (d) $\frac{a^2 - 4b^2}{8a}$

45. $\frac{r_1}{r_2}$ equals

(a) $\frac{a}{b}\left(\frac{4b^2 + a^2}{4a^2 + b^2}\right)$ (b) $\frac{b}{a}\left(\frac{4a^2 + b^2}{4b^2 + a^2}\right)$

(c) $\frac{a}{b}\left(\frac{4b^2-a^2}{4a^2-b^2}\right)$ (d) $\frac{b}{a}\left(\frac{a^2-4b^2}{4a^2-b^2}\right)$

46. Minimum value of $(r_1 + r_2)$ equals

(a) $\frac{5}{8}(a-b)$ (b) $\frac{5}{8}(a+b)$

(c) $\frac{3}{8}(a-b)$ (d) $\frac{3}{8}(a+b)$

Matrix-Match Type

47. Match the following :

	Column I		Column II
A.	If a circle passes through $A(1, 0)$ $B(0, -1)$ and $C\left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$ such that the tangent at B makes an angle θ with line AB then $\tan\theta$ equals	p.	-4
B.	From a point $(h, 0)$ common tangents are drawn to the circles $x^2 + y^2 = 1$ and the $(x-2)^2 + y^2 = 4$. The value of h can be	q.	-2
C.	If the common chord of the circle $x^2 + y^2 = 8$ and $(x-a)^2 + y^2 = 8$ subtends right angle at the origin then a can be	r.	1
D.	If the tangents drawn from $(4, k)$ to the circle $x^2 + y^2 = 10$ are at right angles then k can be	s.	2
		t.	4

Integer Answer Type

48. Line segment AC and BD are diameters of circle of radius one. If $\angle BDC = 60°$, the length of line segment AB is

49. The radius of the circles which pass through the point (2, 3) and cut off equal chords of length 6 units along the lines $y - x - 1 = 0$ and $y + x - 5 = 0$ is 'r' then $[r]$ is (where $[\cdot]$ denotes greatest integer function)

50. Let $M(-1, 2)$ and $N(1, 4)$ be two points in a plane rectangular coordinate system XOY. P is a moving point on the x-axis. When $\angle MPN$ takes its maximum value, the x-coordinate of point P is

51. The equation of the tangent at the point $\left(\frac{ab^2}{a^2+b^2}, \frac{a^2b}{a^2+b^2}\right)$ of the circle $x^2+y^2=\frac{a^2b^2}{a^2+b^2}$ is $\frac{x}{a}+\frac{y}{b}=\lambda$, then λ is

52. The line $y = x + c$ will intersect the circle $x^2 + y^2 = 1$ in two coincident points, if then c^2 equals

53. The locus of the point of intersection of the tangents at the extremities of a chord of the circle $x^2+y^2=a^2$ which touches the circle $x^2+y^2=2ax$ is given by $y^2 = k(a - 2x)a$, then, k is

54. The two circles which passes through $(0, a)$ and and $(0, -a)$ touch the line $y = mx + c$ will intersect each other at right angle, then $\frac{c^2-a^2m^2}{a^2}$ equals

55. The equation of the tangent to the circle $x^2 + y^2 = a^2$ which makes a triangle of area a^2 with the coordinate axes, is $x \pm y = ak$, then k^2 is

SOLUTIONS

1. (b) : Let the centre of the required circle be (x_1, y_1) and the centre of given circle is (1, 2). Since radii of both circles are same, therefore, point of contact (5, 5) is the mid point of the line joining the centres of both circles. Hence $x_1 = 9$ and $y_1 = 8$. Hence the required equation is $(x-9)^2 + (y-8)^2 = 25$

$\Rightarrow x^2 + y^2 - 18x - 16y + 120 = 0.$

2. (b) : Centre $\equiv$ (1, 2) and since circle touches x-axis, therefore, radius of the circle is 2.

Hence the equation is $(x-1)^2 + (y-2)^2 = 2^2$

$\Rightarrow x^2 + y^2 - 2x - 4y + 1 = 0.$

3. (c) : $2\sqrt{g^2-c} = 2a$ ...(i)

$2\sqrt{f^2-c} = 2b$ ...(ii)

On squaring (i) and (ii) and then subtracting (ii) from (i), we get $g^2 - f^2 = a^2 - b^2$.

Hence the locus is $x^2 - y^2 = a^2 - b^2$.

4. (b) : The diameter of the circle is perpendicular distance between the parallel lines (tangents) $3x - 4y + 4 = 0$ and $3x-4y-\frac{7}{2}=0$ and so it is equal to $\frac{|4+7/2|}{\sqrt{9+16}}=\frac{3}{2}$.

Hence radius is $\frac{3}{4}$.

5. (b) : Let us find the equation of family of circles through (2, −2) and (−1, −1).

i.e., $(x-2)(x+1)+(y+2)(y+1)+\lambda\left(\frac{y+2}{-2+1}-\frac{x-2}{2+1}\right)=0$

Now for point (5, 2) to lie on it,

$3\cdot6+4\cdot3+\lambda\left(\frac{4}{-1}-1\right)=0 \Rightarrow \lambda=\frac{30}{5}=6$

Hence required equation is

$$(x-2)(x+1)+(y+2)(y+1)+6\left(\frac{y+2}{-1}-\frac{x-2}{3}\right)=0$$

or $x^2+y^2-3x-3y-8=0.$

6. (c) : The equation of circle in third quadrant touching the coordinate axes with centre $(-a, -a)$ and radius 'a' is $x^2+y^2+2ax+2ay+a^2=0$ and we know

$$\left|\frac{3(-a)-4(-a)+8}{\sqrt{9+16}}\right|=a\Rightarrow a=2$$

Hence the required equation is

$x^2+y^2+4x+4y+4=0.$

7. (a) : The point of intersection of $3x+y-14=0$ and $2x+5y-18=0$ is $(4, 2)$.

Therefore radius is $\sqrt{9+16}=5$ and equation of the circle is $x^2+y^2-2x+4y-20=0$.

8. (b) : Let centre of circle be (h, k). Since it touches both axes, therefore $h=k=a$.

Hence equation can be $(x-a)^2+(y-a)^2=a^2$

But it also touches the line $3x+4y=4$.

Therefore, $\left|\frac{3a+4a-4}{5}\right|=a\Rightarrow a=2$

Hence the required equation of circle is

$x^2+y^2-4x-4y+4=0$

9. (d) : If the circle $x^2+y^2+2gx+2fy+c=0$ touches the x-axis, then $|f|=\sqrt{g^2+f^2-c}\Rightarrow g^2=c$ ...(i)

and cuts a chord of length $2l$ from y-axis

$\Rightarrow 2\sqrt{f^2-c}=2l\Rightarrow f^2-c=l^2$ ...(ii)

Subtracting (i) from (ii), we get $f^2-g^2=l^2$.

Hence the locus is $y^2-x^2=l^2$, which is obviously a hyperbola.

10. (b) : Let centre of the circle be (h, k), then

$\sqrt{(h-2)^2+(k-3)^2}=\sqrt{(h-4)^2+(k-5)^2}$ ...(i)

and $k-4h+3=0$ ...(ii)

From (i), we get $h+k-7=0$ ...(iii)

From (ii) and (iii), we get (h, k) as $(2, 5)$. Hence centre is $(2, 5)$ and radius is 2. Now the equation of circle is

$(x-2)^2+(y-5)^2=4$

$\Rightarrow x^2+y^2-4x-10y+25=0$

11. (a) : Equation of circle concentric to given circle is

$x^2+y^2-6x+12y+k=0$ ...(i)

Radius of circle (i) $=\sqrt{2}$ (radius of given circle)

$\Rightarrow \sqrt{9+36-k}=\sqrt{2}\sqrt{9+36-15}$

$\Rightarrow 45-k=60\Rightarrow k=-15$

Hence the required equation of circle is

$x^2+y^2-6x+12y-15=0$

12. (b) : We have the equation of circle

$x^2+y^2+2gx+2fy+c=0$

But it passes through $(0, 0)$ and $(2, 1)$, then

$c=0$ ....(i)

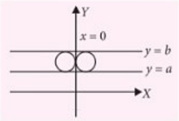
$5+4g+2f=0$ ....(ii)

Also $\sqrt{g^2+f^2-c}=|g|\Rightarrow f=0\ \{\because c=0\}$

From (ii), $g=-\frac{5}{4}$

Hence the equation will be $2x^2+2y^2-5x=0$.

13. (b) : There are only two circles as shown in figure.



14. (b) : Let a triangle has its three vertices as $(0, 0)$, $(a, 0)$, $(0, b)$. We have the moving point (h, k) such that

$h^2+k^2+(h-a)^2+k^2+h^2+(k-b)^2=c$

$\Rightarrow 3h^2+3k^2-2ah-2bk+a^2+b^2=c$

Therefore, $3x^2+3y^2-2ax-2by+a^2+b^2=c$

Its centre is $\left(\frac{a}{3},\frac{b}{3}\right)$, which is centroid of triangle.

15. (b) : Given that centre of circle is $(2, 2)$.

Since circle is passing through $(4, 5)$.

So, radius of circle

$$=\sqrt{(4-2)^2+(5-2)^2}=\sqrt{4+9}=\sqrt{13}$$

Therefore, equation of circle is

$$(x-2)^2+(y-2)^2=(\sqrt{13})^2$$

$\Rightarrow x^2+y^2-4x-4y-5=0$

16. (b) : Let (h, k)be the centre of the circle which rolls on the outside of the given circle. Centre of the given circle is $\left(\frac{-3}{2},3\right)$ and its radius $=\sqrt{\frac{9}{4}+9+9}=\frac{9}{2}$.

Clearly, (h, k) is always at a distance equal to the $\left(\frac{9}{2}+2\right)=\frac{13}{2}$ of the radii of two circles from $\left(-\frac{3}{2},3\right)$.

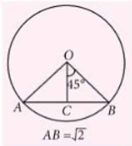
Therefore $\left(h+\frac{3}{2}\right)^2+(k-3)^2=\left(\frac{13}{2}\right)^2$

$\Rightarrow h^2+k^2+3h-6k+\frac{9}{4}+9-\frac{169}{4}=0$

Hence locus of (h, k) is $x^2+y^2+3x-6y-31=0$.

17. (c) : Let AB be the chord of length $\sqrt{2}$, O be centre of the circle and let OC be the perpendicular from O on AB. Then,

$AC = BC = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$



In $\Delta OBC, OB = BC \operatorname{cosec} 45^o$

$= \frac{1}{\sqrt{2}} . \sqrt{2} = 1$

Area of the circle $= \pi(OB)^2$

$= \pi$ sq. units.

18. (b) : Let the equation of line through the point (α, β) be $\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta} = k$ (say) ...(i)

where k is the distance of any point (x, y) on the line from the point $P(\alpha, \beta)$. Let this line meets the circle $x^2 + y^2 = r^2$ at $(\alpha + k\cos\theta, \beta + k\sin\theta)$.

$\therefore \quad (\alpha + k\cos\theta)^2 + (\beta + k\sin\theta)^2 = r^2$

or $\quad k^2 + 2(\alpha\cos\theta + \beta\sin\theta)k + (\alpha^2 + \beta^2 - r^2) = 0,$

which is a quadratic in k. If k_1 and k_2 are its roots and the line (i) meets circle at A and B, then $PA = k_1$ and $PB = k_2$.

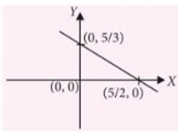
$\therefore \quad PA \cdot PB = k_1 k_2 =$ Products of roots $= \alpha^2 + \beta^2 - r^2$.

19. (b) : Given,triangle formed by the lines $x = 0$, $y = 0$, $2x + 3y = 5$, so vertices of the triangle are $(0, 0)$, $(5/2, 0)$ and $(0, 5/3)$.

Since circle is passing through $(0, 0)$.

$\therefore$ Equation of circle will be

$x^2 + y^2 + 2gx + 2fy = 0$ ...(i)



Also, circle is passing through $(5/2, 0)$ and $(0, 5/3)$

So, $g = -5/4, f = -5/6$.

Put the values of g and f in equation (i), we get $6(x^2 + y^2) - 5(3x + 2y) = 0$, which is the required equation of the circle.

20. (a) : Let point (x_1, y_1) on the diameter.

$\Rightarrow \quad 2x_1 + 3y_1 = 3$ ...(i)

and $16x_1 - y_1 = 4$ ...(ii)

On solving (i) and (ii), we get centre,

$\Rightarrow \quad x_1 = \frac{3}{10}, y_1 = \frac{4}{5}$

Since, circle passes through $(4, 6)$

So, $r^2 = \left(\frac{37}{10}\right)^2 + \left(\frac{26}{5}\right)^2 \Rightarrow r^2 = \frac{4073}{100}$

$\therefore$ Required equation of circle is

$\left(x - \frac{3}{10}\right)^2 + \left(y - \frac{4}{5}\right)^2 = \frac{4073}{100}$

$\Rightarrow \quad 5(x^2 + y^2) - 3x - 8y = 200.$

21. (b) : Required equation of the circle,

$(x^2 + y^2 - 2x - 6y + 6) + \lambda(3x + 2y - 5) = 0$

This circle passing through points $(-2, 4)$, therefore

$(4 + 16 + 4 - 24 + 6) + \lambda(-6 + 8 - 5) = 0, \ \therefore \ \lambda = 2$

$\therefore \quad (x^2 + y^2 - 2x - 6y + 6) + 2(3x + 2y - 5) = 0$

$\Rightarrow \quad x^2 + y^2 + 4x - 2y - 4 = 0$

22. (b) : $x = 2 + 3\cos\theta, y = 3\sin\theta - 1$

$x^2 + y^2 = 4 + 9\cos^2\theta + 12\cos\theta + 9\sin^2\theta + 1 - 6\sin\theta$

$= 14 + 12\cos\theta - 6\sin\theta$

$= 4(2 + 3\cos\theta) - 2(3\sin\theta - 1) + 4$

$\Rightarrow \quad x^2 + y^2 = 4x - 2y + 4$

$\Rightarrow \quad (x^2 - 4x + 4) + (y^2 + 2y + 1) = 9$

$\Rightarrow \quad (x - 2)^2 + (y + 1)^2 = 9$, $\therefore$ Centre is $(2, -1)$.

23. (d) : General equation of circle is,

$x^2 + y^2 + 2gx + 2fy + c = 0$

It passes through $(0,0)$, $(1, 0)$ and $(0, 1)$; $\therefore \ c = 0$

Now $2g + 1 = 0 \Rightarrow g = -\frac{1}{2}$ and $2f + 1 = 0 \Rightarrow f = \frac{-1}{2}$

Hence equation of circle is

$x^2 + y^2 - x - y = 0$

Since, point $(2k, 3k)$ lies on the circle

$\therefore \quad 4k^2 + 9k^2 - 5k = 0$

$\Rightarrow \quad 13k^2 - 5k = 0 \ \Rightarrow \ k = 0$ or $k = \frac{5}{13}$

24. (c) : The other end is $(t, 3 - t)$

So the equation of the variable circle is

$(x - 1)(x - t) + (y - 1)(y - 3 + t) = 0$

or $\quad x^2 + y^2 - (1 + t)x - (4 - t)y + 3 = 0$

$\therefore \quad$ The centre (α, β) is given by

$\alpha = \frac{1+t}{2}, \ \beta = \frac{4-t}{2}$

$\Rightarrow \quad 2\alpha + 2\beta = 5$

Hence, the locus is $2x + 2y = 5$.

25. (d) : $x\cos\alpha + y\sin\alpha - p = 0$ is a tangent, if perpendicular from centre on it is equal to radius of the circle. Here centre is $(a\cos\alpha, a\sin\alpha)$ and radius is a.

$\therefore \quad \left|\frac{a\cos^2\alpha + a\sin^2\alpha - p}{\sqrt{1}}\right| = a$

i.e. $|a - p| = a \Rightarrow p = 0$ or $p = 2a$.

26. (b) : The equation of tangents is $SS_1 = T^2$

$\Rightarrow \quad h^2(x^2 + y^2 - 2rx - 2hy + h^2) = (rx + hy - h^2)^2$

$\Rightarrow \quad (h^2 - r^2)x^2 - 2rhxy = 0 \Rightarrow x\{(h^2 - r^2)x - 2rhy\} = 0$

$\Rightarrow \quad x = 0, (h^2 - r^2)x - 2rhy = 0$

27. (b) : If the line $lx + my - 1$ touches the circle $x^2 + y^2 = a^2$, then applying the condition of tangency, we have $\frac{|l \cdot 0 + m \cdot 0 - 1|}{\sqrt{l^2 + m^2}} = a$

On squaring and simplifying, we get the required locus $x^2+y^2=\frac{1}{a^2}$. Hence it is a circle.

28. (a) : Equation of line parallel to the $\sqrt{3}x+y+3=0$ is $\sqrt{3}x+y+k=0$

But it is a tangent to the circle $x^2 + y^2 = a^2$, then

$$\left|\frac{k}{\sqrt{1+3}}\right|=a\Rightarrow k=\pm 2a$$

$\therefore$ Equation of tengent to the circle is

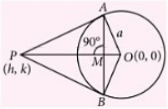
$$\sqrt{3}x+y\pm 2a=0$$

29. (a) : Equation of chord of contact AB is

$xh + yk = a^2$ ...(i)

OM = length of perpendicular from $O(0, 0)$ on line (i)

$=\frac{a^2}{\sqrt{h^2+k^2}}$



$\therefore\ AB=2AM=2\sqrt{OA^2-OM^2}=\frac{2a\sqrt{h^2+k^2-a^2}}{\sqrt{h^2+k^2}}$

Also PM = length of perpendicular from $P(h, k)$ to the line (i) is $\frac{h^2+k^2-a^2}{\sqrt{h^2+k^2}}$

Therefore, the required area of triangle PAB

$$=\frac{1}{2}\cdot AB\cdot PM=\frac{a(h^2+k^2-a^2)^{3/2}}{h^2+k^2}$$

30. (a) : The abscissa of point is found by substituting the ordinates and solving for abscissa.

$\Rightarrow\ x^2 - 8x + 15 = 0 \Rightarrow x = 5$ or 3

i.e., Points are $(5, -1)$ and $(3, -1)$.

Normal is given by, $\frac{x-5}{5-4}=\frac{y+1}{-1-1}\Rightarrow 2x+y-9=0$

and $\frac{x-3}{3-4}=\frac{y+1}{-1-1}\Rightarrow 2x-y-7=0$

31. (a) : The equation of tangents will be

$c(x^2 + y^2 + 2gx + 2fy + c) = (gx + fy + c)^2$

These tangents are perpendicular, hence the coefficients of x^2 + coefficients of $y^2 = 0$

$\Rightarrow\ c - g^2 + c - f^2 = 0\ \Rightarrow\ f^2 + g^2 = 2c.$

32. (a) : Tangent is $x - 2y - 5 = 0$ and points of intersection with circle $x^2 + y^2 - 8x + 6y + 20 = 0$ are given by

$4y^2 + 25 + 20y + y^2 - 16y - 40 + 6y + 20 = 0$

$\Rightarrow\ 5y^2 + 10y + 5 = 0$

$\Rightarrow\ y = -1$ and $x = 3$ *i.e.*, touches.

33. (c) : Substituting equation of line $y = mx + c$ in circle $x^2 + y^2 = r^2$

$\therefore\ x^2 + (mx + c)^2 = r^2 \Rightarrow (1 + m)^2x^2 + 2mxc + c^2 - r^2 = 0$ If discriminant is greater than zero; two real values of x will be obtained.

So, $B^2 > 4AC$

$\Rightarrow\ 4m^2c^2 - 4(c^2 - r^2)(1 + m^2) > 0 \Rightarrow r^2(1 + m^2) > c^2$

$\Rightarrow\ 0\le c<r\sqrt{1+m^2}$ and $-r\sqrt{(1+m^2)}<c\le 0$

34. (a, b) : $16l^2 + 9m^2 = 24lm + 6l + 8m + 1$

$\Rightarrow\ 25(l^2 + m^2) = 9l^2 + 16m^2 + 24lm + 6l + 8m + 1$

$= (3l + 4m + 1)^2$

$\Rightarrow\ \frac{|l(3)+m(4)+1|}{\sqrt{l^2+m^2}}=5$

Centre = $(3, 4)$, radius = 5

35. (a, b, c) : Given circle is $x^2 + y^2 = a^2$ ... (i)

Clearly $(0, 0)$ will belong to the interior of circle (i)

Also other points interior to circle (i) will have the co-ordinates of the form $(\pm\alpha, 0)$, $(0, \pm\alpha)$ where $\alpha^2 < a^2$ and $(\pm\alpha, \pm\beta)$, $(\pm\beta, \pm\alpha)$, where $\alpha^2 + \beta^2 < a^2$ and $\alpha, \beta \in I$.

$\therefore$ Number of lattice points in the interior of the circle will be of the form $1 + 4k + 8r$,

Where $k, r = 0, 1, 2, \ldots$..

Number of such points must be of the form $4m + 1$, where $m = 0, 1, 2, \ldots$...

36. (a, d) : Tangent $\Rightarrow\ \lambda = 15, -35$

$\lambda = 15\ \Rightarrow\ (a, b) = (5, 0)$

$\lambda = -35\ \Rightarrow\ (a, b) = (-1, 8)$

37. (a, b, c, d) : $\because\ \angle A=\alpha$ is a constant. If 'I' is incentre of ΔABC.

$\angle BIC=90^\circ+\frac{\alpha}{2}$ which is fixed.

Hence 'I' lies on a fixed circle of which BC is a fixed chord.

$\because\ \angle A=\alpha$. If '$H$' is orthocentre then $\angle BHC = 180^\circ - \alpha$ which is fixed.

Hence, 'H' lies on a circle of which BC is fixed chord.

$\therefore\ \angle A = \alpha$, If '$G$' is centroid of the triangle. $\angle TGK = \alpha$ where T, K are points of trisection of base BC which are fixed.

The fixed line segment TK subtends a constant angle α at a variable point G.

Hence, locus of centroid is also lies on circle.

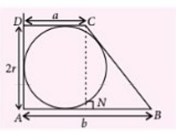
38. (b, c) : $DC + AB = AD + CB \Rightarrow CB = a + b - 2r$

The triangle CNB gives

$(2r)^2 + (b - a)^2 = (a + b - 2r)^2$

Which gives $r=\frac{ab}{a+b}$

$\Rightarrow 2r=\frac{2ab}{a+b}$



Area $= \frac{1}{2}(a+b)2r = ab$

39. (c, d) : The given circles will cut orthogonally if

$$2\left(\frac{p}{2}\right)\left(\frac{-10}{2}\right)+2\left(\frac{p}{2}\right)\left(\frac{2p}{2}\right)=-7+1$$

or $p^2 - 5p + 6 = 0 \Rightarrow p = 2$ or 3.

40. (a, c) : Orthogonal tangents to a circle meet at the director circle

$\therefore$ $S_2 \equiv x^2 + y^2 = 2\cdot 1 \Rightarrow S_2 \equiv x^2 + y^2 = 2$

Also, $S_3 \equiv x^2 + y^2 = 4$

Ratio of radius of S_2 and $S_3 = \sqrt{2}:2 = 1:\sqrt{2}$

Also, the three circles are concentric.

41. (c) : The equation of the coaxal system is $x^2 + y^2 + 4x + 2y + 5 + \lambda(x^2 + y^2 + 2x + 4y + 7) = 0$

or $x^2+y^2+\frac{2(2+\lambda)}{1+\lambda}x+\frac{2(1+2\lambda)}{1+\lambda}y+\frac{5+7\lambda}{1+\lambda}=0$

Equating radius to zero, we get

$$\frac{(2+\lambda)^2+(1+2\lambda)^2-(5+7\lambda)(1+\lambda)}{(1+\lambda)^2}=0$$

$\Rightarrow$ $2\lambda^2 + 4\lambda = 0 \Rightarrow \lambda = 0$ or -2

The centre of above system is $\left(-\frac{2+\lambda}{1+\lambda},-\frac{1+2\lambda}{1+\lambda}\right)$

Substituting the values of λ, we get the Coordinates of limiting points $(-2, -1)$ and $(0, -3)$

42. (d) : The point circles represented by the limiting points are $(x - 1)^2 + (y + 1)^2 = 0$ and $(x - 2)^2 + y^2 = 0$

So, the equation of coaxal system is

$(x - 1)^2 + (y + 1)^2 + \lambda\{(x - 2)^2 + y^2\} = 0$ ..... (i)

It passes through (0, 0) so, $\lambda = -\frac{1}{2}$

Putting the value of λ in (i) we get the equation to the desired circle as $x^2 + y^2 + 4y = 0$.

43. (b) : The equation of the given coaxal system is $x^2 + y^2 - 2\alpha x - 2\beta y + c + \lambda(x^2 + y^2) = 0$

or $x^2+y^2-\frac{2\alpha}{1+\lambda}x-\frac{2\beta}{1+\lambda}y+\frac{c}{1+\lambda}=0$

Its centre is $\left(\frac{\alpha}{1+\lambda},\frac{\beta}{1+\lambda}\right)$ and radius is

$$\frac{\sqrt{\alpha^2+\beta^2-c(1+\lambda)}}{|1+\lambda|}$$

The radius vanishes if $1+\lambda=\frac{\alpha^2+\beta^2}{c}$

So, the other limiting point is $\left(\frac{c\alpha}{\alpha^2+\beta^2},\frac{c\beta}{\alpha^2+\beta^2}\right)$.

(44-46) :

44. (b) **45. (a)** **46. (b)**

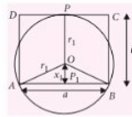
Let $r_1 = b - x_1 = OP = OA$

$\therefore$ $AP_1 = a/2$

$r_1^2 = x_1^2 + (a/2)^2 = (b - x_1)^2$

$\therefore$ $x_1^2+\frac{a^2}{4}=b^2+x_1^2-2bx_1 \Rightarrow x_1=\frac{4b^2-a^2}{8b}$

$\Rightarrow$ $r_1=b-x_1=\frac{4b^2+a^2}{8b}$



Similarly for the circle passing through B and C and touching side AD, $r_2=\frac{4a^2+b^2}{8a}$

Now, $r_1+r_2=\frac{4b^2+a^2}{8b}+\frac{4a^2+b^2}{8a}$

$=\frac{a^3+b^3+4ab(a+b)}{8ab}=\frac{(a+b)(a^2+3ab+b^2)}{8ab}$

$=\frac{(a+b)}{8}\cdot\frac{(a^2-2ab+b^2+5ab)}{ab}=\frac{(a+b)}{8}\cdot\frac{[(a-b)^2+5ab]}{ab}$

But $(a - b)^2 \geq 0$

$\therefore$ $r_1+r_2\geq\frac{(a+b)}{8}\cdot\frac{5ab}{ab}\Rightarrow r_1+r_2\geq\frac{5(a+b)}{8}$

47. A → r; B → q; C → p, t; D → q, s

A. Origin is the circumcentre

$\Rightarrow$ Circle is $x^2+y^2=1 \Rightarrow \theta=\frac{\pi}{4}$

B. A tangent to $x^2 + y^2 = 1$ is $y=mx\pm\sqrt{1+m^2}$. It touches $(x-2)^2+y^2=4$ if $\left|\frac{2m\pm\sqrt{1+m^2}}{\sqrt{1+m^2}}\right|=2$

$\Rightarrow$ $m=\pm\frac{1}{\sqrt{3}}$

$\therefore$ The common tangents are $y=\frac{1}{\sqrt{3}}x+\frac{2}{\sqrt{3}}$ and $y=-\frac{1}{\sqrt{3}}x-\frac{2}{\sqrt{3}}$ which intersect at $(-2, 0)$.

C. Common chord of the given circles is $(x^2 + y^2 - 8) - [(x - a)^2 + y^2 - 8] = 0 \Rightarrow 2x - a = 0$

$\Rightarrow$ $\frac{2x}{a}=1$

Since, $x^2+y^2-8=0\Rightarrow x^2+y^2-8\left(\frac{2x}{a}\right)^2=0$

It represents perpendicular lines

$\Rightarrow$ $1-\frac{32}{a^2}+1=0 \Rightarrow a^2=16 \Rightarrow a=\pm 4$

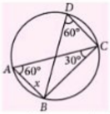
D. $(4, k)$ must lie on the director circle of the given circle which is $x^2 + y^2 = 20$. Thus

$16 + k^2 = 20 \Rightarrow k = \pm 2$

48. (1) : $\angle A = 60° = \angle D$

$AC = 2$(given), $\angle ABC = 90°$

$\Rightarrow \quad x = 1$



49. (4) : The given two lines pass through the point (2, 3) and are inclined at 45° and 135° to the x–axis. The other ends of chords can easily be calculated as $(2+3\sqrt{2}, 3+3\sqrt{2})$ and $(2-3\sqrt{2}, 3-3\sqrt{2})$

There is symmetry about the line $x = 2$ and therefore the centres of circles lie on $x = 2$.

As the chords subtend right angles at the centre.

$\therefore \; 2r^2 = 6^2 \Rightarrow \; r = 3\sqrt{2}$

50. (1) : The centre of a circle passing through points M and N lies on the perpendicular bisector $y = 3 - x$ of MN. Denote the centre by $C(a, 3 - a)$, the equation of the circle is $(x - a)^2 + (y - 3 + a)^2 = 2(1 + a^2)$

Since for a chord with a fixed length the angle at the circumference subtended by the corresponding arc will become larger as the radius of the circle becomes smaller. When $\angle MPN$ reaches its maximum value the circle through the three points M, N and P will be tangent to the x-axis at P, which means

$2(1 + a^2) = (a - 3)^2 \Rightarrow a = 1$ or $a = -7$

Thus the point of contact are $P(1, 0)$ or $P'(-7, 0)$ respectively.

But the radius of circle through the points M, N and P' is larger than that of circle through points M, N and P. Therefore, $\angle MPN > \angle MP'N$. Thus $P = (1, 0)$

$\therefore$ x-coordinate of $P = 1$.

51. (1) : From formula of tangent at a point,

$$x\left(\frac{ab^2}{a^2+b^2}\right)+y\left(\frac{a^2b}{a^2+b^2}\right)=\frac{a^2b^2}{a^2+b^2} \Rightarrow \frac{x}{a}+\frac{y}{b}=1$$

52. (2) : $y = x + c$ intersects at two coincident points, *i.e.*, It is a tangent, therefore $c = \pm\sqrt{2}$.

53. (1) : $T \equiv hx + ky - a^2 = 0$

$$\Rightarrow \; a = \frac{ah+0-a^2}{\sqrt{h^2+k^2}}$$

$\Rightarrow \; h^2 + k^2 = (h - a)^2 \Rightarrow k^2 = a(a - 2h)$

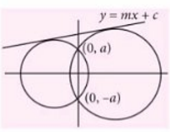
$\therefore$ The locus is $y^2 = a(a - 2x)$.

54. (2) : Equation of circles

$[x^2 + (y - a)(y + a)] + \lambda x = 0$

$\Rightarrow x^2 + y^2 + \lambda x - a^2 = 0$

and $\sqrt{\left(\frac{\lambda}{2}\right)^2 + a^2} = \dfrac{\frac{-m\lambda}{2}+c}{\sqrt{1+m^2}}$



$$\Rightarrow (1+m^2)\left[\frac{\lambda^2}{4}+a^2\right]=\left(\frac{m\lambda}{2}-c\right)^2$$

$$\Rightarrow (1+m^2)\left[\frac{\lambda^2}{4}+a^2\right]=\frac{m^2\lambda^2}{4}-mc\lambda+c^2$$

$\Rightarrow \; \lambda^2 + 4mc\lambda + 4a^2(1 + m^2) - 4c^2 = 0$

$\therefore \; \lambda_1\lambda_2 = 4[a^2(1 + m^2) - c^2]$

$\Rightarrow \; g_1g_2 = [a^2(1 + m^2) - c^2]$

and $g_1g_2 + f_1f_2 = \dfrac{c_1+c_2}{2} \Rightarrow a^2(1+m^2)-c^2 = -a^2$

Hence, $\dfrac{c^2-a^2m^2}{a^2}=2$

55. (2) : Let the tangent be of form $\dfrac{x}{x_1}+\dfrac{y}{y_1}=1$ and area of Δ formed by it with coordinate axes is

$\frac{1}{2}x_1y_1 = a^2$ ....(i)

Again, $y_1x + x_1y - x_1y_1 = 0$

Applying conditions of tangency

$\left|\dfrac{-x_1y_1}{\sqrt{x_1^2+y_1^2}}\right| = a$ or $(x_1^2+y_1^2)=\dfrac{x_1^2y_1^2}{a^2}$ ....(ii)

From (i) and (ii), we get $x_1 \cdot y_1$; which gives equation of tangent as $x \pm y = \pm a\sqrt{2}$.

CBSE warm-up!

CLASS-XI

Synopsis and Chapterwise Practice questions for CBSE Exams as per the latest pattern and marking scheme issued by CBSE for the academic session 2020-21.

Series 1 — Set, Relations and Functions

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) This question paper contains **36** questions.
(iii) Question **1-20** in Section-A are very short-answer type questions carrying **1** mark each.
(iv) Question **21-26** in Section-B are short-answer type questions carrying **2** marks each.
(v) Question **27-32** in Section-C are long-answer-**I** type questions carrying **4** marks each.
(vi) Question **33-36** in Section-D are long-answer-**II** type questions carrying **6** marks each.

SECTION - A

(Q.1 - Q.10) are multiple choice type questions. Select the correct option.

1. If the ordered pairs $(x - 1, y + 3)$ and $(2, x + 4)$ are equal, the value of x and y respectively are
(a) 4, 3 (b) 3, 4 (c) 5, 2 (d) 2, 5

2. If $A \times B = \{(p, q), (p, r), (m, q), (m, r)\}$, then sets A and B respectively are
(a) $\{p, m\}, \{q, r\}$ (b) $\{p, r\}, \{m, q\}$
(c) $\{p, q\}, \{r, m\}$ (d) $\{p, r\}, \{q, m\}$

3. The set builder form of the set {3, 6, 9, 12, ...} is
(a) $\{x : x = 3n, n \in N\}$
(b) $\{x : x = n + 3, n \in N\}$
(c) $\{x : x = n + 2, n \in N\}$
(d) $\{x : x = 3n + 2, n \in N\}$

4. The number of subsets of a set having 'n' element is
(a) 2^{n^2} (b) 2^n (c) 2^{n+1} (d) 2^{n+2}

5. If $A = \{-1, 0, 1, 2\}$, then $n(P(A))$ equal to
(a) 16 (b) 8 (c) 32 (d) 15

6. If A and B are two sets, then $A \cap (A - B)$ is equal to
(a) A (b) B
(c) ϕ (d) None of these

7. If A and B are two sets then $(A \cup B')' \cap (A' \cup B)'$ is
(a) Null set (b) Universal set
(c) A' (d) B'

8. Let $n(A) = 8$ and $n(B) = p$. Then, the total number of non-empty relations that can be defined from A to B is
(a) 8^p (b) $n^p - 1$ (c) $8p - 1$ (d) $2^{8p} - 1$

9. Let $f(x) = 1 + x$, $g(x) = x^2 + x + 1$, then $(f + g)(x)$ at $x = 0$ is
(a) 2 (b) 5 (c) 6 (d) 9

10. If $n(A) = 12$, $n(B) = 8$, $n(A \cap B) = 4$, then $n(A \cup B)$ is equal to
(a) 20 (b) 12 (c) 16 (d) 18

(Q.11-Q.15) Fill in the blanks.

11. If A and B each having k elements common to both then $A \times B$ and $B \times A$ have ________ elements in common.

12. For all sets A and B, $A - (A \cap B)$ is equal to ______.

OR

If X and Y are two sets and X' denotes the complement of X, then $X \cap (X \cup Y)$is equal to ______.

13. Let R be the relation on the set N of natural numbers defined by $R = \{(a, b) : a + 3b = 12, a \in N, b \in N\}$. Then domain of R = ______.

14. If $f(x + 1) = 3x + 5$, then $f(x)$ = ______.

OR

Let $A = \{x, y, z\}$ and $B = \{1, 2\}$, the number of relations from A into B is ______.

15. If A and B are two sets given in such a way that $A \times B$ consists of 6 elements and if three elements of $A \times B$ are $(1, 3)$, $(2, 5)$, $(3, 3)$, then remaining elements of $A \times B$ are ______.

(Q.16-Q.20) Answer the following questions.

16. Are the following sets equal?
$A = \{x : x$ is a letter in the word 'wolf'$\}$
$B = \{x : x$ is a letter in the word 'follow'$\}$
$C = \{x : x$ is a letter in the word 'flow'$\}$

17. Write the following set in the set builder form :
$Q = \{1, 4, 9, 16\}$

OR

List all the elements of sets:
(i) $C = \{x : x = 2n, n \in N$ and $n \le 5\}$
(ii) $H = \{x : x = n^2, n \in N, 2 \le n \le 5\}$

18. Let $f : R \to R$ is defined by $f(x) = \cos x, \forall x \in R$. Is f a function on R? If yes, then find its range.

19. If $f(x) = \dfrac{2\tan x}{1+\tan^2 x}$, then find $f\left(\dfrac{\pi}{4}\right)$.

20. Write the set $X = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, ...\right\}$ in the set-builder form.

SECTION - B

21. Given $f(x) = \dfrac{1}{(1-x)}$, $g(x) = f\{f(x)\}$ and $h(x) = f[f\{f(x)\}]$. Then find the value of $f(x) \cdot g(x) \cdot h(x)$.

22. Let A and B be two sets. Prove that :
$(A - B) \cup B = A$ if and only if $B \subseteq A$

OR

If $A = \{3, 5, 7, 9, 11\}$, $B = \{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$ and $D = \{15, 17\}$, find:
(i) $A \cap (B \cup C)$ (ii) $(A \cup D) \cap (B \cup C)$

23. If $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$, prove that $f\left(\dfrac{2x}{1+x^2}\right) = 2f(x)$.

24. A survey shows that 73% of the Indians like apples, whereas 65% like oranges. What percentage of Indians like both apples and oranges?

25. Let $f : R \to R$ be defined by $f(x) = \sin x$ and $g : R \to R$ be defined by $g(x) = x$, find $f + g, f - g, f \cdot g, \dfrac{f}{g}$ and $\alpha f, \alpha \in R$.

OR

Find the domain of the function $f(x)$ given by
$$f(x) = \frac{1}{\sqrt{[x]^2 - [x] - 6}}.$$

26. Let A, B and C be three sets such that $A \cup B = C$ and $A \cap B = \phi$, then prove that $A = C - B$.

SECTION - C

27. If R is a relation given by
$$R = \left\{(x, y) : y = x + \frac{6}{x} \text{ where } x, y \in N \text{ and } x < 6\right\}$$
Find the following :
(a) Ray diagram (b) R in roster form
(c) Domain of R (d) Range of R

28. If $A = \{1, 2, 3, 5, 9\}$, $B = \{2, 4, 6, 8, 9, 11\}$, $C = \{1, 3, 6, 8, 10\}$ and $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, then find
(i) $(A - B) \cap C'$ (ii) $A' - (B - C)$
(iii) $A - (B \cup C)'$ (iv) $(A \cup B)' \cap C$

OR

Draw the Venn diagram to represent the following sets:
(i) $(A \cup B) - C$ (ii) $A - (B \cap C)$
(iii) $(A \cup B) \cap C$ (iv) $(A \cap B) - C$

29. If $f(x) = \dfrac{1+x}{1-x}$, prove that $\dfrac{f(x) \cdot f(x^2)}{1 + [f(x)]^2} = \dfrac{1}{2}$

30. In a survey of 400 students in a school, 100 were listed as drinking apple juice, 150 as drinking orange juice and 75 were listed as drinking apple as well as orange juice. Find how many students were drinking neither apple juice nor orange juice?

31. Find the domain and range of the function
$$f(x) = \frac{1}{2 - \sin 3x}$$

OR

A function $f: R \rightarrow R$ is defined as

$$f(x)=\begin{cases}2x+3, & x \geq 3\\ 7, & x<3\end{cases}, \text{ find}$$

(i) $f(1)$ (ii) $f(-4)$ (iii) $f(6)$ (iv) $f(3)$

32. Prove that $A \cap (B - C) = (A \cap B) - (A \cap C)$

SECTION - D

33. A school awarded 15 medals in table tennis, 38 in hockey and 20 in cricket. If these medals went to a total of 58 men and only three men got medals in all the three sports, how many received medals in exactly two of the three sports?

34. If $A = \{1, 2, 5\}$, $B = \{1, 2, 3, 4\}$ and $C = \{5, 6, 2\}$, then verify that :

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(ii) $(A - B) \times C = (A \times C) - (B \times C)$.

OR

Find the domain and range of the following real valued functions :

(a) $f(x)=\dfrac{1}{\sqrt{x^2-81}}$ (b) $f(x)=\dfrac{2x}{6-x}$

35. There are 2000 students in a school. Out of these 1000 play cricket, 600 play basketball and 550 play football, 120 play cricket and basketball, 80 play basketball and football, 150 play cricket and football and 45 play all the three games. How many students play none of the games?

OR

Draw the Venn diagrams to illustrate the following relationship among sets, E, M and U, where E is the set of students studying English in a school, M is the set of students studying Mathematics in the same school, U is the set of all students in that school.

(i) All the students who study Mathematics study English, but some students who study English do not study Mathematics.

(ii) There is no student who studies both Mathematics and English.

(iii) Some of the students study Mathematics but do not study English, some study English but do not study Mathematics, and some study both.

(iv) Not all students study Mathematics, but every student studying English studies Mathematics.

36. If $f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$ then find the domain for which

(a) $f(x) = g(x)$ (b) $f(x) - g(x) = 3$

(c) $f(x) + g(x) = 2$

SOLUTIONS

1. **(b) :** Given, $(x - 1, y + 3) = (2, x + 4)$

$\Rightarrow x - 1 = 2$ and $y + 3 = x + 4$

$\Rightarrow x = 3$ and $y = x + 1 \Rightarrow x = 3$ and $y = 3 + 1 = 4$

Hence, $x = 3$ and $y = 4$

2. **(a) :** A = set of first elements = $\{p, m\}$

B = set of second elements = $\{q, r\}$.

3. **(a) :** Let $A = \{3, 6, 9, 12, ...\}$

All elements of the set are natural numbers that are multiples of 3.

$\therefore$ $A = \{x : x = 3n, \ n \in N\}$

4. **(b) :** Number of subsets of a set having 'n' elements is 2^n.

5. **(a) :** Since, $n(A) = 4$. Therefore, $n(P(A)) = 2^4 = 16$

6. **(d) :** $A \cap (A - B) = A - B$

7. **(a) :** Consider $(A \cup B')' \cap (A' \cup B)'$

$= (A' \cap B) \cap (A \cap B') = (B - A) \cap (A - B) = \phi$

8. **(d) :** Given $n(A) = 8$ and $n(B) = p$

$\therefore$ Total number of relations from A to $B = 2^{8p}$

$\therefore$ Total number of non-empty relations from A to $B = 2^{8p} - 1$

9. **(a) :** We have, $f(x) = 1 + x$ and $g(x) = x^2 + x + 1$

$\therefore$ $(f + g)(x) = f(x) + g(x)$

$= 1 + x + x^2 + x + 1 = x^2 + 2x + 2$

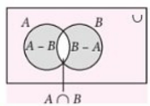
$\therefore$ $(f + g)(0) = (0)^2 + 2(0) + 2 = 2$

10. **(c) :** $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 12 + 8 - 4 = 16.$

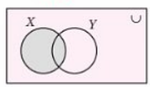
11. If $n(A \cap B) = k$, then $A \times B$ and $B \times A$ have k^2 elements in common.

12. $A - (A \cap B) = A - B$



OR

$X \cap (X \cup Y) = X$



13. $R = \{(a, b) : a = 12 - 3b, a \in N, b \in N\}$

$= \{(9, 1), (6, 2), (3, 3)\}$

$\therefore$ Domain of $R = \{9, 6, 3\}$

14. Given, $f(x + 1) = 3x + 5$

Putting $x - 1$ in place of x, we get

$f(x - 1 + 1) = 3(x - 1) + 5$

$\Rightarrow f(x) = 3x + 2$

OR

Given, $A = \{x, y, z\}$ and $B = \{1, 2\}$

$\therefore\ n(A) = 3$ and $n(B) = 2$

$\therefore\ n(A \times B) = n(A) \cdot n(B) = 3 \cdot 2 = 6$

Total number of relations from A into B

$= $ number of subsets of $A \times B = 2^6 = 64$.

15. Since $(1, 3), (2, 5), (3, 3) \in A \times B$, so clearly $1, 2, 3 \in A$ and $3, 5 \in B$

Given, $n(A \times B) = 6 \Rightarrow n(A) \cdot n(B) = 6$

But $1, 2, 3 \in A$ and $3, 5 \in B$, therefore $n(A) = 3$ and $n(B) = 2$

$\therefore\ A = \{1, 2, 3\}$ and $B = \{3, 5\}$

$\therefore\ A \times B = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)\}$

$\therefore$ Remaining elements of $A \times B$ are $(1, 5)$, $(2, 3)$ and $(3, 5)$.

16. $A = \{w, o, l, f\}$, $B = \{f, o, l, w\}$, $C = \{f, l, o, w\}$

Clearly, $A = B = C$.

17. Here, $Q = \{1, 4, 9, 16\}$.

Note that each number in the set is a perfect square.

$\therefore\ Q = \{x : x$ is square of a natural number and $x \le 16\}$

$\Rightarrow Q = \{x : x$ is square of a natural number less than $5\}$

OR

(i) $C = \{2, 4, 6, 8, 10\}$

(ii) $H = \{4, 9, 16, 25\}$

18. Since for each $x \in R$, $\cos x$ is unique real number, therefore f is a function.

Since, $-1 \le \cos x \le 1$, so range of $f = [-1, 1]$.

19. We have, $f(x) = \dfrac{2\tan x}{1+\tan^2 x}$

$$\therefore f\left(\frac{\pi}{4}\right) = \frac{2\tan\frac{\pi}{4}}{1+\tan^2\frac{\pi}{4}} = \frac{2\cdot 1}{1+1} = \frac{2}{2} = 1.$$

20. The elements of set X are the reciprocals of the squares of all natural numbers. So, the set X in set builder form is $X = \left\{\dfrac{1}{n^2} : n \in N\right\}$.

21. Given, $g(x) = f\{f(x)\}$

$$= f\left(\frac{1}{1-x}\right) = \frac{1}{1-\frac{1}{1-x}} = -\left(\frac{1-x}{x}\right)$$

and $h(x) = f[f\{f(x)\}] = f\{g(x)\} = f\left(-\left(\dfrac{1-x}{x}\right)\right)$

$$= \frac{1}{1+\frac{1-x}{x}} = x$$

$$\therefore\ f(x)\cdot g(x)\cdot h(x) = \frac{1}{1-x}\left\{-\left(\frac{1-x}{x}\right)\right\}\cdot x = -1$$

22. First, let $(A - B) \cup B = A$. Then, we have to prove that $B \subseteq A$.

Now, $(A - B) \cup B = A$

$\Rightarrow (A \cap B') \cup B = A$ $\quad [\because A - B = A \cap B']$

$\Rightarrow (A \cup B) \cap (B' \cup B) = A$

$\Rightarrow (A \cup B) \cap U = A$

$\Rightarrow A \cup B = A \Rightarrow B \subseteq A$

Conversely, let $B \subseteq A$. Then, we have to prove that $(A - B) \cup B = A$.

Now, $(A - B) \cup B = (A \cap B') \cup B$

$= (A \cup B) \cap (B' \cup B) = (A \cup B) \cap U$

$= A \cap U = A$ $\quad [\because B \subseteq A \therefore A \cup B = A]$

OR

(i) $A \cap (B \cup C)$

$= \{3, 5, 7, 9, 11\} \cap (\{7, 9, 11, 13\} \cup \{11, 13, 15\})$

$= \{3, 5, 7, 9, 11\} \cap \{7, 9, 11, 13, 15\} = \{7, 9, 11\}$

(ii) $(A \cup D) \cap (B \cup C)$

$= (\{3, 5, 7, 9, 11\} \cup \{15, 17\}) \cap (\{7, 9, 11, 13\} \cup \{11, 13, 15\})$

$= \{3, 5, 7, 9, 11, 15, 17\} \cap \{7, 9, 11, 13, 15\} = \{7, 9, 11, 15\}$

23. Given, $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$

$$\therefore\ f\left(\frac{2x}{1+x^2}\right) = \log\left(\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right) = \log\left(\frac{1+x^2+2x}{1+x^2-2x}\right)$$

$$= \log\left(\frac{1+x}{1-x}\right)^2 = 2\log\left(\frac{1+x}{1-x}\right) = 2f(x).$$

24. Let A = Set of Indians who like apples and B = Set of Indians who like oranges.
Then, $n(A) = 73$, $n(B) = 65$ and $n(A \cup B) = 100$
Now, $n(A \cap B) = n(A) + n(B) - n(A \cup B)$
$= 73 + 65 - 100 = 38$.
Hence, 38% of the Indians like both apples and oranges.

25. Given, $f(x) = \sin x$ and $g(x) = x$
Now, $f + g : R \to R$, $(f + g)(x) = f(x) + g(x) = \sin x + x$
$(f - g) : R \to R$, $(f - g)(x) = f(x) - g(x) = \sin x - x$
$(f \cdot g) : R \to R$, $(f \cdot g)(x) = f(x) \cdot g(x) = (\sin x)x = x \sin x$

$\frac{f}{g}: R \to R$, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sin x}{x}, x \neq 0$

$\alpha f : R \to R$, $(\alpha f)(x) = \alpha f(x) = \alpha \sin x$

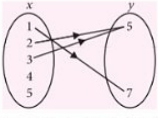
OR

Given, $f(x) = \frac{1}{\sqrt{[x]^2 - [x] - 6}}$.

For D_f, $f(x)$ must be a real number
$\Rightarrow [x]^2 - [x] - 6 > 0 \Rightarrow ([x] + 2)([x] - 3) > 0$
$\Rightarrow [x] < -2$ or $[x] > 3 \Rightarrow x < -2$, or $x \geq 4$
$\Rightarrow D_f = (-\infty, -2) \cup [4, \infty)$.

26. Here $A \cup B = C$
$\therefore (A \cup B) - B = C - B$
$\Rightarrow (A \cup B) \cap B' = C - B$ $\quad (\because A - B = A \cap B')$
$\Rightarrow (A \cap B') \cup (B \cap B') = C - B$
$\Rightarrow (A \cap B') \cup \phi = C - B$
$\Rightarrow (A \cap B') = C - B$
$\Rightarrow A = C - B$ $\quad (\because A \cap B = \phi)$

27. Given, $R = \{(x, y) : y = x + \frac{6}{x}$, where $x, y \in N$ and $x < 6\} = \{(1, 7), (2, 5), (3, 5)\}$

(a)


(b) $R = \{(1, 7), (2, 5), (3, 5)\}$
(c) Domain of $R = \{1, 2, 3\}$
(d) Range of $R = \{7, 5\}$

28. (i) $A - B = A - (A \cap B) = \{1, 3, 5\}$
$C' = \{2, 4, 5, 7, 9, 11, 12\}$
$(A - B) \cap C' = \{5\}$
(ii) $A' = \{4, 6, 7, 8, 10, 11, 12\}$
$B - C = B - (B \cap C) = \{2, 4, 9, 11\}$
$A' - (B - C) = \{6, 7, 8, 10, 12\}$
(iii) $B \cup C = \{1, 2, 3, 4, 6, 8, 9, 10, 11\}$
$(B \cup C)' = \{5, 7, 12\}$
$A - (B \cup C)' = \{1, 2, 3, 9\}$
(iv) $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 9, 11\}$
$(A \cup B)' = \{7, 10, 12\}$
$(A \cup B)' \cap C = \{10\}$

OR

Refer to answer 62, Page no. 14 of MTG CBSE Champion Mathematics, Class-11

29. $\because f(x) = \frac{1+x}{1-x} \quad \therefore f(x^2) = \frac{1+x^2}{1-x^2}$

Now, $\frac{f(x) \cdot f(x^2)}{1 + [f(x)]^2} = \frac{\left(\frac{1+x}{1-x}\right)\left(\frac{1+x^2}{1-x^2}\right)}{1 + \left(\frac{1+x}{1-x}\right)^2}$

$$= \frac{\frac{1+x^2}{(1-x)^2}}{\frac{(1-x)^2 + (1+x)^2}{(1-x)^2}} = \frac{1+x^2}{2(1+x^2)} = \frac{1}{2}$$

30. Let U = Set of students surveyed.
A = Set of people drinking apple juice.
B = Set of people drinking orange juice.
Given, $n(U) = 400$, $n(A) = 100$,
$n(B) = 150$ and $n(A \cap B) = 75$
Now $n(A' \cap B') = n(A \cup B)'$
$= n(U) - n(A \cup B)$
$= n(U) - [n(A) + n(B) - n(A \cap B)]$
$= 400 - 100 - 150 + 75 = 225$.
$\therefore$ Number of students drinking neither apple juice nor orange juice is 225.

31. We have, $f(x) = \frac{1}{2 - \sin 3x}$

Domain of f: We know that $-1 \leq \sin 3x \leq 1$ for all $x \in R$
$\Rightarrow -1 \leq -\sin 3x \leq 1$ for all $x \in R$
$\Rightarrow 1 \leq 2 - \sin 3x \leq 3$ for all $x \in R$
$\Rightarrow 2 - \sin 3x \neq 0$ for any $x \in R$
$\Rightarrow f(x) = \frac{1}{2 - \sin 3x}$ is defined for all $x \in R$

Hence, domain $(f) = R$.
Range of f: $\because 1 \leq 2 - \sin 3x \leq 3$ for all $x \in R$

$\Rightarrow \frac{1}{3} \leq \frac{1}{2 - \sin 3x} \leq 1$ for all $x \in R$

$\Rightarrow \frac{1}{3} \leq f(x) \leq 1$ for all $x \in R$

$\Rightarrow f(x) \in [1/3, 1]$

Hence, range $(f) = [1/3, 1]$

OR

Refer to answer 102, Page no. 38 of MTG CBSE Champion Mathematics, Class-11

32. Let $x \in A \cap (B - C)$

$\Rightarrow x \in A$ and $x \in (B - C)$

$\Rightarrow x \in A$ and $(x \in B$ and $x \notin C)$

$\Rightarrow (x \in A$ and $x \in B)$ and $(x \in A$ and $x \notin C)$

$\Rightarrow x \in (A \cap B)$ and $x \notin (A \cap C)$

$\Rightarrow x \in (A \cap B) - (A \cap C)$

Hence $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$ ...(i)

Now, let $y \in (A \cap B) - (A \cap C)$

$\Rightarrow y \in (A \cap B)$ and $y \notin (A \cap C)$

$\Rightarrow (y \in A$ and $y \in B)$ and $(y \in A$ and $y \notin C)$

$\Rightarrow y \in A$ and $(y \in B$ and $y \notin C)$

$\Rightarrow y \in A$ and $y \in (B - C)$

$\Rightarrow y \in A \cap (B - C)$

$\therefore (A \cap B) - (A \cap C) \subseteq A \cap (B - C)$ ...(ii)

$\therefore$ From (i) and (ii), we have

$A \cap (B - C) = (A \cap B) - (A \cap C)$

33. Let T denote the set of men who received medals in table tennis, B the set of men who received medals in hockey and C the set of men who received medals in cricket. Then, we have

$n(T) = 15, n(H) = 38, n(C) = 20, n(T \cup H \cup C) = 58$

and $n(T \cap H \cap C) = 3$

Now, $n(T \cup H \cup C) = n(T) + n(H) + n(C) - n(T \cap H) - n(H \cap C) - n(T \cap C) + n(T \cap H \cap C)$

$\Rightarrow 58 = 38 + 15 + 20 - n(T \cap H) - n(H \cap C) - n(T \cap C) + 3$

$\Rightarrow n(T \cap H) + n(H \cap C) + n(T \cap C) = 76 - 58 = 18$

Now, number of men who received medals in exactly two of the three sports

$= n(T \cap H) + n(H \cap C) + n(T \cap C) - 3n(T \cap H \cap C) = 18 - 3 \times 3 = 9$

Thus, 9 men received medals in exactly two of the three sports.

34. Given, $A = \{1, 2, 5\}, B = \{1, 2, 3, 4\}$ and $C = \{5, 6, 2\}$

(i) $B \cap C = \{2\}$

L.H.S. $= A \times (B \cap C) = \{1, 2, 5\} \times \{2\}$

$= \{(1, 2), (2, 2), (5, 2)\}$

Now, $A \times B = \{1, 2, 5\} \times \{1, 2, 3, 4\}$

$= \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (5,1), (5,2), (5,3), (5,4)\}$

and $A \times C = \{1, 2, 5\} \times \{5, 6, 2\}$

$= \{(1, 5), (1, 6), (1, 2), (2, 5), (2, 6), (2, 2), (5, 5), (5, 6), (5, 2)\}$

$\therefore$ R.H.S. $= (A \times B) \cap (A \times C)$

$= \{(1, 2), (2, 2), (5, 2)\} =$ L.H.S.

(ii) $A - B = \{5\}$

L.H.S. $= (A - B) \times C = \{5\} \times \{5, 6, 2\}$

$= \{(5, 5), (5, 6), (5, 2)\}$

Now, $A \times C = \{1, 2, 5\} \times \{5, 6, 2\}$

$= \{(1, 5), (1, 6), (1, 2), (2, 5), (2, 6), (2, 2), (5, 5), (5, 6), (5, 2)\}$

and $B \times C = \{1, 2, 3, 4\} \times \{5, 6, 2\}$

$= \{(1, 5), (1, 6), (1, 2), (2, 5), (2, 6), (2, 2), (3, 5), (3, 6), (3, 2), (4, 5), (4, 6), (4, 2)\}$

$\therefore$ R.H.S. $= (A \times C) - (B \times C)$

$= \{(5, 5), (5, 6), (5, 2)\} =$ L.H.S.

OR

Refer to answer 131, Page no. 41 of MTG CBSE Champion Mathematics, Class-11

35. A = Set of students playing cricket

B = Set of students playing basketball

C = Set of students playing football

Now, $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

$= 1000 + 600 + 550 - 120 - 80 - 150 + 45 = 1845$

Number of students playing none of the games

$= 2000 - 1845 = 155.$

OR

Refer to answer 67, Page no. 15 of MTG CBSE Champion Mathematics, Class-11

36. Given, $f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$

(a) $f(x) = g(x)$

$\Rightarrow 2x^2 - 1 = 1 - 3x \Rightarrow 2x^2 + 3x - 2 = 0$

$\Rightarrow (x + 2)(2x - 1) = 0 \Rightarrow x = -2, \frac{1}{2}$

$\therefore$ Required domain $= \left\{-2, \frac{1}{2}\right\}$

(b) $f(x) - g(x) = 3$

$\Rightarrow 2x^2 - 1 - 1 + 3x = 3 \Rightarrow 2x^2 + 3x - 5 = 0$

$\Rightarrow (x - 1)(2x + 5) = 0 \Rightarrow x = 1, -\frac{5}{2}$

$\therefore$ Required domain $= \left\{-\frac{5}{2}, 1\right\}$

(c) $f(x) + g(x) = 2$

$\Rightarrow 2x^2 - 1 + 1 - 3x = 2 \Rightarrow 2x^2 - 3x - 2 = 0$

$\Rightarrow (x - 2)(2x + 1) = 0$

$\Rightarrow x = 2, -\frac{1}{2}$

$\therefore$ Required domain $= \left\{2, -\frac{1}{2}\right\}$

MONTHLY TEST DRIVE

XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80

Series 1 : Sets, Relations and Functions

Time Taken : 60 Min.

Only One Option Correct Type

1. If $X = \{8^n - 7n - 1 : n \in N\}$ and $Y = \{49(n - 1) : n \in N\}$, then

(a) $X \subset Y$ (b) $Y \subset X$
(c) $X = Y$ (d) none of these

2. 20 teachers of a school either teach Mathematics or Physics. 12 of them Mathematics while 4 teach both the subjects. The number of teachers teaching Physics only, is

(a) 12 (b) 8 (c) 16 (d) 18

3. Let A and B be two sets containing 2 elements and 4 elements respectively. The number of subsets of $A \times B$ having 3 or more elements is

(a) 256 (b) 220 (c) 219 (d) 211

4. If P is the set of all parallelograms and T is the set of all trapeziums, then $P \cap T$ is

(a) P (b) T
(c) ϕ (d) none of these

5. Let Z denotes the set of integers, then
$\{X \in Z : |x - 3| < 4\} \cap \{X \in Z : |x - 4| < 5\} =$

(a) $\{-1, 0, 1, 2, 3, 4\}$ (b) $\{-1, 0, 1, 2, 3, 4, 5\}$
(c) $\{0, 1, 2, 3, 4, 5, 6\}$ (d) $\{-1, 0, 1, 2, 3, 5, 6, 7, 8, 9\}$

One or More Than One Option(s) Correct Type

6. If domain of f is D_1 and domain of g is D_2, then domain of $f + g$ is

(a) D_1/D_2 (b) $D_1 - (D_1/D_2)$
(c) $D_2/(D_2/D_1)$ (d) $D_1 \cap D_2$

7. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then

(a) $A = C$ (b) $B = C$
(c) $A \cap B = \phi$ (d) $A = B$

8. If $f(x) = \left(\dfrac{x-1}{x+1}\right)$, then which of the following statement(s) is/are correct?

(a) $f\left(\dfrac{1}{x}\right) = f(x)$ (b) $f\left(\dfrac{1}{x}\right) = -f(x)$

(c) $f\left(-\dfrac{1}{x}\right) = \dfrac{1}{f(x)}$ (d) $f\left(-\dfrac{1}{x}\right) = -\dfrac{1}{f(x)}$

9. If A_n is the set of first n prime numbers, then $\bigcup_{n=2}^{10} A_n =$

(a) $\{2, 3, 5, 7, 11, 13, 17, 19\}$
(b) $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$
(c) $\{3, 5\}$
(d) $\{2, 3\}$

10. Which of the following functions are not identical?

(a) $f(x) = \dfrac{x}{x^2}$ and $g(x) = \dfrac{1}{x}$

(b) $f(x) = \dfrac{x^2}{x}$ and $g(x) = x$

(c) $f(x) = \ln x^4$ and $g(x) = 4 \ln x$
(d) $f(x) = \ln\{(x - 1)(x - 2)\}$ and $g(x) = \ln(x - 2) + \ln(x - 3)$

11. If $A = \left\{n : \dfrac{n^3 + 5n^2 + 2}{n} \text{ is an integer}\right\}$, then the number of elements in the set A, is

(a) 1 (b) 2
(c) 3 (d) 4

12. If $y = f(x) = \dfrac{x+2}{x-1}$, then

(a) $x = f(y)$ (b) $f(1) = 3$
(c) $f(2) = 4$ (d) f is rational function of x

13. If $e^x + e^{f(x)} = e$, then for $f(x)$

(a) domain $= (-\infty, 1)$ (b) range $= (-\infty, 1)$
(c) domain $= (-\infty, 0]$ (d) range $= (-\infty, 1]$

Comprehension Type

Let f be a function satisfying $f(x)=\dfrac{a^x}{a^x+\sqrt{a}}=g_a(x), a>0.$

14. Let $f(x) = g_4(x)$, then $\sum_{r=1}^{1996} f\left(\dfrac{r}{1997}\right)$ is

(a) 0 (b) 1997 (c) 998 (d) 1

15. The value of $g_5(x) + g_5(1 - x)$ is

(a) 1 (b) 10 (c) 5 (d) 6

Matrix Match Type

16. Match the following:

	Column-I		Column-II
P.	If $y = 2\|x - 2\| + 3\|x + 1\|$, then	1.	$y=\begin{cases}-5x+1, & x<-1\\ x+7, & -1\le x<2\end{cases}$
Q.	If $y = 2\|x - 2\| - 3\|x + 1\|$, then	2.	$y=\begin{cases}5x-1, & -1\le x<2\\ x+7, & x\ge 2\end{cases}$
R.	If $y = 3\|x + 1\| - 2\|x - 2\|$, then	3.	$y=\begin{cases}x+7, & x<-1\\ -5x+1, & -1\le x<2\end{cases}$
		4.	$y=\begin{cases}x+7, & -1\le x<2\\ 5x-1, & x\ge 2\end{cases}$

	P	Q	R
(a)	1, 4	2	3
(b)	1, 3	1, 4	2
(c)	1, 4	3	2
(d)	2	3	1, 3

Numerical Value Type

17. If f is a function such that $f(0) = 2, f(1) = 3$ and $f(x + 2) = 2f(x) - f(x + 1)$ for every real x, then $f(5)$ = ____.

18. If $A = \{1, 2, 3, 4\}$, then the number of subsets of set A containing element 3, is ____.

19. If A and B are two sets such that $n(A\cap\bar{B})=9$, $n(\bar{A}\cap B)=10$ and $n(A \cup B) = 24$, then $n(A \times B)$ = ____.

20. If $P = \{a, b, c\}$ and $Q = \{1, 2\}$, then the total number of relations from P to Q which are not functions, is ____.

◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……

No. of questions correct ……

Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CONCEPT MAP

COMPLEX NUMBERS

Class XI

DEFINITION

Number of the form
$z = a + ib$ where $a, b \in R$,
a = Real part and
b = Imaginary part

BASIC TERMS

For complex number $z = x + iy$

- **Conjugate :** $\bar{z} = x - iy$
- **Modulus :** $|z| = \sqrt{x^2 + y^2}$
- **Argument :** $\tan^{-1}\left(\frac{y}{x}\right)$

Properties

For $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

- $\overline{z_1 + z_2} = \bar{z}_1 \pm \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- $(z^{\overline{n}}) = (\bar{z})^n$
- $|z| = |\bar{z}| = |-z|$
- $z\bar{z} = |z|^2$
- $|z^n| = |z|^n$
- $|z_1 + z_2 + z_3 + ... + z_n| \le |z_1| + |z_2| + ... + |z_n|$
- $\arg|z_1 z_2| = \arg z_1 + \arg z_2$
- $\arg|z_1 / z_2| = \arg z_1 - \arg z_2$
- $\arg|z^n| = n\arg(z)$.

SOME FACTS ABOUT LOCUS

If z is a variable point and z_1, z_2 are two fixed points in the Argand plane :

- $|z - z_1| = |z - z_2|$ represents perpendicular bisector of the segment joining z_1 and z_2.
- $|z - z_1| + |z - z_2| = K$ (a fixed quantity > 0) ...(i)
  - If $K > |z_1 - z_2|$ then (i) represents an ellipse.
  - If $K = |z_1 - z_2|$ then (i) represents the segment joining z_1 and z_2.
  - If $K < |z_1 - z_2|$ then (i) does not represent any curve in the Argand plane.
- $|z - z_1| - |z - z_2| = K(> 0)$
  - If $K \ne |z_1 - z_2|$ then $|z - z_1| - |z - z_2| = K$ represent a hyperbola with foci at z_1 and z_2.
  - If $K = |z_1 - z_2|$ then $|z - z_1| - |z - z_2| = K$ represents a straight line joining z_1 and z_2 but excluding the segment joining z_1 and z_2.
- $|z - z_1|^2 + |z - z_2|^2 = K = |z_1 - z_2|^2$ represent a circle with affixes z_1 and z_2 the extremities of a diameter and $K \ge \frac{1}{2}|z_1 - z_2|^2$
- $|z - z_1| = K|z - z_2|$, $(K \ne 1)$, then locus of z is a circle, *i.e.* $\left|\frac{z - z_1}{z - z_2}\right| = K\ (K \ne 0)$ represent a circle, for $K = 1$, represent a straight line.
- $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$ (a fixed quantity) then locus of z is a circle.
- $\arg\left(\frac{z - z_1}{z - z_2}\right) = \pm\frac{\pi}{2}$, then locus of z is a circle as z_1 and z_2 are vertices of the end point of the diameter.
- $\arg\left(\frac{z - z_1}{z - z_2}\right) = 0$ or π then locus of z is a line passing through the points z_1 and z_2.

GEOMETRICAL APPLICATIONS

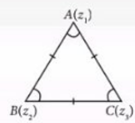
Triangle

Triangle ABC with vertices $A(z_1)$, $B(z_2)$, $C(z_3)$ is equilateral if and only if

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

$$\Leftrightarrow z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

$$\Leftrightarrow \begin{vmatrix} 1 & z_1 & z_2 \\ 1 & z_2 & z_3 \\ 1 & z_3 & z_1 \end{vmatrix} = 0$$



Circle

- The equation of circle whose centre is at point having affix z_0 and radius R is $|z - z_0| = R$.
  or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + z_0\bar{z}_0 - R^2 = 0$

$\Rightarrow z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ where $a = -z_0$ and $b = |z_0|^2 - R^2$

$\Rightarrow z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ represents a circle having centre $-a$ and radius $R = \sqrt{|a|^2 - b}$.

CONTINUITY

Class XII

CONCEPT MAP

CONTINUITY AT A POINT

- A function $f(x)$ is said to be continuous at a point $x = a$ in its domain iff $\lim_{x\to a} f(x)$ exist finitely, $f(a)$ is a finite number and $\lim_{x\to a} f(x) = f(a)$.
- **In open interval :** In an open interval (a, b), $f(x)$ is continuous if it is continuous at every point between (a, b).
- **In closed interval :** In a closed interval $[a, b]$, $f(x)$ is continuous if
  - $f(x)$ is continuous in (a, b)
  - $\lim_{x\to a^+} f(x) = f(a)$, $\lim_{x\to b^-} f(x) = f(b)$
- **Continuity everywhere :** A function is said to be continuous everywhere if it is continuous on the entire real number line $(-\infty, \infty)$.

DISCONTINUITY AT A POINT

A function $f(x)$ which is not continuous at point say $(x = a)$, then it is discontinuous at $x = a$.

Types

- **Removable discontinuity :** Discontinuity at $x = a$ $\lim_{x\to a} f(x) \neq f(a)$. It is called removable because it can be made continuous by redefining it at point a.
- **Discontinuity of 1st kind :** $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$
- **Discontinuity of 2nd kind :** $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+}$ or both do not exist.

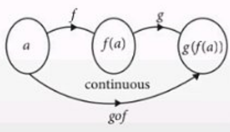
ALGEBRA ON CONTINUOUS FUNCTIONS

Let $f(x)$ & $g(x)$ be two continuous functions on their common domain D & α be any real number then

- $\alpha f(x)$ is continuous
- $f \pm g$ is continuous
- fg is continuous
- f/g is continuous provided $g(x) \neq 0$ for any $x \in D$
- $f^n(x), \forall\, n \in N$ is continuous.

RESULTS

- Identity function, Modulus function, Constant function, Exponential function, Logarithmic function, Polynomial functions are continuous.
- The largest (greatest) integer function $[x]$ is continuous at all points except at integer (integral) points.
- All trigonometric function are continuous in their respective domains like $\sin x$, $\cos x$ are continuous $\forall\, x\, R$.
- If f, g are continuous functions, then fog & gof are continuous. If f is continuous at a point $x = a$ & g is continuous at $f(a)$, then gof is continuous at $x = a$.



CONCEPT BOOSTERS

Class XII

Matrices and Determinants

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR

MATRICES

DEFINITION

A rectangular arrangement of numbers (which may be real or complex numbers) in rows and columns, is called a matrix. This arrangement is enclosed within small ( ) or big [ ] brackets. The numbers are called the elements or entries of the matrix.

ORDER OF A MATRIX

A matrix having m rows and n columns is called a matrix of order $m \times n$.

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & ...a_{1j} & ...a_{1n} \\ a_{21} & a_{22} & a_{23} & ...a_{2j} & ...a_{2n} \\ ..... & ..... & ..... & ..... & ..... \\ a_{i1} & a_{i2} & a_{i3} & ...a_{ij} & ...a_{in} \\ ..... & ..... & ..... & ..... & ..... \\ a_{m1} & a_{m2} & a_{m3} & ..a_{mj} & ...a_{mn} \end{bmatrix}_{m\times n} \text{ or } A=[a_{ij}]_{m\times n},$$

where $i=1, 2,....., m$ and $j=1, 2,....., n$.

TYPES OF MATRICES

(1) **Row matrix :** A matrix is said to be a row matrix if it has only one row and any number of columns.

(2) **Column matrix :** A matrix is said to be a column matrix if it has only one column and any number of rows.

(3) **Singleton matrix :** If in a matrix there is only one element then it is called singleton matrix.
$A=[a_{ij}]_{m\times n}$ is a singleton matrix, if $m=n=1$

(4) **Null or zero matrix :** If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by O.
$A=[a_{ij}]_{m\times n}$ is a zero matrix if $a_{ij}=0$ for all i and j.

(5) **Square matrix:** If number of rows and number of columns in a matrix are equal, then it is called a square matrix.
$A=[a_{ij}]_{m\times n}$ is a square matrix if $m=n$.

(6) **Rectangular matrix :** If in a matrix number of rows and number of columns are not equal, then it is called rectangular matrix.
$A=[a_{ij}]_{m\times n}$ is a rectangular matrix if $m \neq n$.

(7) **Diagonal matrix :** If all elements except the principal diagonal in a square matrix are zero, it is called a diagonal matrix. Thus a square matrix $A=[a_{ij}]$ is a diagonal matrix if $a_{ij}=0$, when $i \neq j$.

(8) **Equal matrices :** Two matrices A and B are said to be equal if and only if they are of same order and their corresponding elements are equal.

(9) **Identity matrix :** A square matrix in which elements in the main diagonal are all '1' and rest are all zero is called an identity matrix or unit matrix. We denote the identity matrix of order n by I_n.

(10) **Scalar matrix :** A square matrix whose all non diagonal elements are zero and diagonal elements are equal is called a scalar matrix.

(11) **Triangular matrix :** A square matrix $[a_{ij}]$ is said to be triangular matrix if each element above or below the principal diagonal is zero. It is of two types :

- **Upper triangular matrix :** A square matrix $[a_{ij}]$ is called the upper triangular matrix, if $a_{ij}=0$ when $i>j$.
- **Lower triangular matrix :** A square matrix $[a_{ij}]$ is called the lower triangular matrix, if $a_{ij}=0$ when $i<j$.

*Alok Kumar, a B.Tech from IIT Kanpur and INMO 4th ranker of his time, has been training IIT and Olympiad aspirants for close to two decades now. His students have bagged AIR 1 in IIT JEE and also won medals for the country at IMO. He has also taught at Maths Olympiad programme at Cornell University, USA and UT, Dallas. He has been regularly proposing problems in international Mathematics journals.

TRACE OF A MATRIX

The sum of diagonal elements of a square matrix A is called the trace of matrix A, which is denoted by $tr\,(A)$.

$$tr\,(A)=\sum_{i=1}^{n} a_{ii}=a_{11}+a_{22}+\ldots+a_{nn}$$

Properties of trace of a matrix

Let $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ and λ be a scalar

- $tr(\lambda A)=\lambda\, tr(A)$
- $tr(A-B)=tr(A)-tr(B)$
- $tr(AB)=tr(BA)$
- $tr(A)=tr(A')$ or $tr(A^T)$
- $tr(I_n)=n$
- $tr(O)=0$
- $tr(AB)\neq tr\,A\cdot tr\,B$

ADDITION AND SUBTRACTION OF MATRICES

If $A=[a_{ij}]_{m\times n}$ and $B=[a_{ij}]_{m\times n}$ are two matrices of the same order then their sum $A+B$ is a matrix whose each element is the sum of corresponding elements *i.e.*, $A=[a_{ij}+b_{ij}]_{m\times n}$.

Similarly, their subtraction $A-B$ is defined as $A-B=[a_{ij}-b_{ij}]_{m\times n}$

Note : Matrix addition and subtraction can be possible only when matrices are of the same order.

Properties of matrix addition

If A, B and C are matrices of same order, then

- $A+B=B+A$ (Commutative law)
- $(A+B)+C=A+(B+C)$ (Associative law)
- $A+O=O+A$, where O is zero matrix which is additive identity of the matrix.
- $A+(-A)=O=(-A)+A$, where $(-A)$ is obtained by changing the sign of every element of A, which is additive inverse of the matrix.
- $\left.\begin{matrix}A+B=A+C\\ B+A=C+A\end{matrix}\right\}\Rightarrow B=C$ (Cancellation law)

SCALAR MULTIPLICATION OF MATRICES

Let $A=[a_{ij}]_{m\times n}$ be a matrix and k be a scalar, then the matrix which is obtained by multiplying every element of A by k is called scalar multiplication of A by k and it is denoted by kA.

Thus, if $A=[a_{ij}]_{m\times n}$, then $kA=Ak=[ka_{ij}]_{m\times n}$.

Properties of scalar multiplication

If A, B are matrices of the same order and λ, μ are any two scalars, then

- $\lambda(A+B)=\lambda A+\lambda B$
- $(\lambda+\mu)A=\lambda A+\mu A$
- $\lambda(\mu A)=(\lambda\mu)A=\mu(\lambda A)$
- $(-\lambda A)=-(\lambda A)=\lambda(-A)$

MULTIPLICATION OF MATRICES

Two matrices A and B are conformable for the product AB if the number of columns in A (pre-multiplier) is same as the number of rows in B (post multiplier). Thus, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ are two matrices of order $m\times n$ and $n\times p$ respectively, then their product AB is of order $m\times p$ and is defined as

$$(AB)_{ij}=\sum_{r=1}^{n} a_{ir}b_{rj}=[a_{i1}a_{i2}...a_{in}]\begin{bmatrix}b_{1j}\\ b_{2j}\\ \vdots\\ b_{nj}\end{bmatrix}$$

$=(i^{\text{th}}$ row of $A)(j^{\text{th}}$ column of $B)$ ...(i)

where $i=1, 2, \ldots, m$ and $j=1, 2, \ldots, p$

Now we define the product of a row matrix and a column matrix.

Let A $=[a_1\, a_2 \ldots\ldots\, a_n]$ be a row matrix and $B=\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}$ be a column matrix.

Then $AB=[a_1b_1+a_2b_2+\ldots..+a_nb_n]$

Thus, from (i), $(AB)_{ij}=$ Sum of the product of elements of i^{th} row A with the corresponding elements of j^{th} column of B.

Properties of matrix multiplication

If A, B and C are three matrices such that their product is defined, then

- $AB\neq BA$ (Commutative law)
- $(AB)C=A(BC)$ (Associative Law)
- $IA=A=AI$, where I is identity matrix of same order
- $A(B+C)=AB+AC$ (Distributive law)
- If $AB=AC \not\Rightarrow B=C$

  (Cancellation law is not applicable)
- If $AB=0$, it does not mean that $A=0$ or $B=0$, again product of two non zero matrix may be a zero matrix.

POSITIVE INTEGRAL POWERS OF A MATRIX

The positive integral powers of a matrix A are defined only when A is a square matrix.

Also then $A^2=A\cdot A$, $A^3=A\cdot A\cdot A=A^2A$

Also for any positive integers m and n,

- $A^mA^n=A^{m+n}$
- $(A^m)^n=A^{mn}=(A^n)^m$
- $I^n=I, I^m=I$
- $A^0=I_n$, where A is a square matrix of order n.

TRANSPOSE OF A MATRIX

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called transpose of matrix A and is denoted by A^T or A'.

Properties of transpose

Let A and B be two matrices of same order $m \times n$ and A^T, B^T be their transpose respectively, then,

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(kA)^T = kA^T$, k be any scalar (real or complex).
- $(AB)^T = B^T A^T$, A and B being conformable for the product AB.
- $(A_1 A_2 A_3 \ldots\ldots A_{n-1} A_n)^T = A_n^T A_{n-1}^T \ldots A_3^T A_2^T A_1^T$
- $I^T = I$

SPECIAL TYPES OF MATRICES

(1) Symmetric matrix : A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ for all i, j or $A^T = A$.

(2) Skew-symmetric matrix : A square matrix $A = [a_{ij}]$ is called skew- symmetric matrix if $a_{ij} = -a_{ji}$ for all i, j or $A^T = -A$.

All principal diagonal elements of a skew- symmetric matrix are always zero because for any diagonal element.
$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

Properties of symmetric and skew-symmetric matrices

- If A is a square matrix, then $A + A^T$, AA^T, A^TA are symmetric matrices, while $A - A^T$ is skew-symmetric matrix.
- If A is a symmetric matrix, then $-A$, kA, A^T, A^n, A^{-1}, B^TAB are also symmetric matrices, where $n \in N$, $k \in R$ and B is a square matrix of order that of A.
- If A is a skew-symmetric matrix, then

(i) A^{2n} is a symmetric matrix for $n \in N$.

(ii) A^{2n+1} is a skew-symmetric matrix for $n \in N$.

(iii) kA is also skew-symmetric matrix, where $k \in R$.

(iv) B^TAB is also skew- symmetric matrix where B is a square matrix of order that of A.

- If A, B are two symmetric matrices, then $A \pm B$, $AB + BA$ are symmetric matrices and $AB - BA$ is a skew- symmetric matrix.
- AB is a symmetric matrix when $AB = BA$.
- If A, B are two skew-symmetric matrices, then $A \pm B$, $AB - BA$ are skew-symmetric matrices and $AB + BA$ is a symmetric matrix.
- If A is a skew-symmetric matrix and C is a column matrix, then C^TAC is a zero matrix.
- Every square matrix A can be uniquely expressed as sum of a symmetric and skew-symmetric matrix

$$\text{i.e., } A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right]$$

(3) Orthogonal matrix : A square matrix A is called orthogonal if $AA^T = I = A^TA$ i.e., if $A^{-1} = A^T$

(4) Idempotent matrix : A square matrix A is called an idempotent matrix if $A^2 = A$.
In fact, every unit matrix is indempotent.

(5) Involutory matrix : A square matrix A is called an involutory matrix if $A^2 = I$ or $A^{-1} = A$.
Every unit matrix is involutory.

(6) Nilpotent matrix : A square matrix A is called a nilpotent matrix, if there exists p, $p \in N$ such that $A^p = O$.

DETERMINANTS

To every square matrix $A = [a_{ij}]_{m \times n}$ is associated a number of function called the determinant of A and denoted by $|A|$.

Consider three homogeneous linear equations
$a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$
and $a_3x + b_3y + c_3z = 0$

Eliminating x, y, z from above three equations, we obtain
$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0$ ...(i)

So, (i) can be also represented by $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

$$\text{i.e., } a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$$

As, it contains three rows and three columns, it is called a determinant of third order.

Properties of determinants

- The value of determinant remains unchanged, if the rows and the columns are interchanged.
- If each element of any row (or column) of determinant can be expressed as a sum of two terms, then the determinant can also be expressed as the sum of two determinants.
- If any two rows (or columns) of a determinant be interchanged, the determinant is unaltered in numerical value but is changed in sign only.
- If all the elements of any row (or column) be multiplied by the same number, then the value of determinant is multiplied by that number.
- If a determinant has two rows (or columns) identical, then its value is zero.
- The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

- If any row (or column) of determinant is multiplied by a non-zero number, then the determinant is also divided by that number.
- If a determinant D becomes zero on putting $x = \alpha$, then we say that $(x - \alpha)$ is factor of determinant.

PRODUCT OF TWO DETERMINANTS

Let the two determinants of third order be,

$$D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Let D be their product.

$$\text{Then, } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

SINGULAR AND NON-SINGULAR MATRIX

Any square matrix A is said to be non-singular if $|A| \neq 0$, and a square matrix A is said to be singular if $|A| = 0$.

CONJUGATE OF A MATRIX

The matrix obtained from any given matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and is denoted by $\bar{A}$.

Properties of conjugate

- $\overline{(\bar{A})} = A$
- $\overline{(A + B)} = \bar{A} + \bar{B}$
- $\overline{(\alpha A)} = \bar{\alpha}\bar{A}$, α being any number
- $\overline{(AB)} = \bar{A}\,\bar{B}$, A and B being conformable for multiplication.

MINOR OF AN ELEMENT

If we take the element of the determinant and delete (remove) the row and column containing that element, the determinant left is called the minor of that element. It is denoted by M_{ij}.

COFACTOR OF AN ELEMENT

The cofactor of an element a_{ij} (*i.e.* the element in the i^{th} row and j^{th} column) is defined as $(-1)^{i+j}$ times the minor of that element. It is denoted by C_{ij}

$C_{ij} = (-1)^{i+j} M_{ij}$

ADJOINT OF A SQUARE MATRIX

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj A.

Thus, adj $A = [C_{ij}]^T$

$\Rightarrow (\text{adj } A)_{ij} = C_{ij}$ = cofactor of a_{ji} in A.

Properties of adjoint matrix

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

- $A(\text{adj } A) = |A| \; I_n = (\text{adj } A)A$
- $|\text{adj } A| = |A|^{n-1}$
- $\text{adj}(\text{adj } A) = |A|^{n-2}A$
- $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$
- $\text{adj}(A^T) = (\text{adj } A)^T$
- $\text{adj }(AB) = (\text{adj } B)(\text{adj } A)$
- $\text{adj}(A^m) = (\text{adj}A)^m, m \in N$
- $\text{adj}(kA) = k^{n-1}(\text{adj}A), k \in R$
- $\text{adj}(I_n) = I_n$

INVERSE OF A MATRIX

A non-singular square matrix A of order n is invertible, if there exists a square matrix B of the same order such that $AB = I_n = BA$.

In such a case, we say that the inverse of A is B and we write $A^{-1} = B$. The inverse of A is given by

$$A^{-1} = \frac{1}{|A|}.\text{adj} A$$

The necessary and sufficient condition for the existence of the inverse of a square matrix A is that $|A| \neq 0$.

Properties of inverse matrix

If A and B are invertible matrices of the same order, then

- $(A^{-1})^{-1} = A$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^k)^{-1} = (A^{-1})^k, k \in N$
- $\text{adj}(A^{-1}) = (\text{adj}A)^{-1}$
- $|A^{-1}| = \frac{1}{|A|}$
- Every invertible matrix possesses a unique inverse.

RANK OF A MATRIX

The rank of a given matrix A is said to be r if

(i) Every minor of A of order $r + 1$ is zero.

(ii) There is atleast one minor of A of order r which does not vanish.

(iii) The rank r of matrix A is written as $\rho(A) = r$.

Rank of a matrix in Echelon form : The rank of a matrix in Echelon form is equal to the number of non-zero rows in that matrix.

HOMOGENEOUS AND NON-HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

A system of equations $AX = B$ is called a homogeneous system if $B = O$. If $B \neq O$, then it is called a non-homogeneous system of equations.

SOLUTION OF NON-HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

Matrix method

If $AX = B$, then $X = A^{-1}B$ gives a unique solution, provided A is non-singular (*i.e.*, $|A| \neq 0$).

But if $|A| = 0$ and $(\text{adj } A).\, B = 0$, then system is consistent with infinitely many solutions otherwise system is incosistent.

SOLUTION OF A HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

- Let $AX = O$ be a homogeneous system of three linear equations in three unknowns.
- Write the given system of equations in the form $AX = O$ and write A.
- Find $|A|$.
- If $|A| \neq 0$, then the system is consistent and $x = y = z = 0$ is the unique solution.
- If $|A| = 0$, then the systems of equations has infinitely many solutions. In order to find these solution put $z = k$ (any real number) and solve any two equations for x and y by matrix method. The values of x and y so obtained with $z = k$ give a solution of the given system of equations.

SOLUTION OF SYSTEM OF LINEAR EQUATIONS IN THREE VARIABLES BY CRAMER'S RULE

- The solution of the system of linear equation given by $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$ is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$,

  where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

  $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

  provided that $D \neq 0$

Conditions for consistency

- If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$
- If $D = 0$ and atleast one of the determinants D_1, D_2, D_3 is non-zero, then given system of equations is inconsistent.

PROBLEMS

Single Correct Answer Type

1. If $a \neq b \neq c$, then the value of x which satisfies the equation $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$, is

(a) $x = 0$ (b) $x = a$ (c) $x = b$ (d) $x = c$

2. If $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0$, then $x =$

(a) 1, 9 (b) − 1, 9 (c) − 1, − 9 (d) 1, − 9

3. $\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} =$

(a) $-2abc$ (b) abc
(c) 0 (d) $a^2 + b^2 + c^2$

4. If $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = k(x+y+z)(x-z)^2$, then $k =$

(a) $2xyz$ (b) 1 (c) xyz (d) $x^2y^2z^2$

5. Value of $\Delta = \begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$ equals

(a) $a^3 + b^3 + c^3 - 3abc$
(b) $3abc - a^3 - b^3 - c^3$
(c) $a^3 + b^3 + c^3 - a^2b - b^2c - c^2a$
(d) $(a + b + c)(a^2 + b^2 + c^2 + ab + bc + ca)$

6. If ω is a complex cube root of unity, then the value of $\begin{vmatrix} 2 & 2\omega & -\omega^2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} =$

(a) 0 (b) 1
(c) − 1 (d) None of these

7. If ω be a complex cube root of unity, then

$$\begin{vmatrix} 1 & \omega & -\omega^2/2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} =$$

(a) 0 (b) 1 (c) ω (d) ω^2

8. $\begin{vmatrix} a_1 & ma_1 & b_1 \\ a_2 & ma_2 & b_2 \\ a_3 & ma_3 & b_3 \end{vmatrix} =$

(a) 0 (b) $ma_1a_2a_3$ (c) $ma_1a_2b_3$ (d) $mb_1a_2a_3$

9. $\begin{vmatrix} 11 & 12 & 13 \\ 12 & 13 & 14 \\ 13 & 14 & 15 \end{vmatrix} =$

(a) 1 (b) 0 (c) –1 (d) 67

10. A root of the equation $\begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0$ is

(a) 6 (b) 3
(c) 0 (d) None of these

11. The roots of the equation

$\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$ are

(a) 1, 2 (b) – 1, 2 (c) 1, – 2 (d) –1, – 2

12. The roots of the determinant equation (in x)

$\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = 0$

(a) $x = a, b$ (b) $x = -a, -b$
(c) $x = -a, b$ (d) $x = a, -b$

13. If a, b, c are in A.P., then the value of

$\begin{vmatrix} x+2 & x+3 & x+a \\ x+4 & x+5 & x+b \\ x+6 & x+7 & x+c \end{vmatrix}$ is

(a) $x - (a + b + c)$ (b) $9x^2 + a + b + c$
(c) $a + b + c$ (d) 0

14. If a (≠ 6), b, c satisfy $\begin{vmatrix} a & 2b & 2c \\ 3 & b & c \\ 4 & a & b \end{vmatrix} = 0$, then $abc =$

(a) $a + b + c$ (b) 0
(c) b^3 (d) $ab + bc$

15. If $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 12 \end{bmatrix}$, then

(a) $AB = O, BA = O$ (b) $AB = O, BA \neq O$
(c) $AB \neq O, BA = O$ (d) $AB \neq O, BA \neq O$

16. If ω is a complex cube root of unity, then

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{vmatrix} =$

(a) $3\sqrt{3}i$ (b) $-2\sqrt{3}i$
(c) $i\sqrt{3}$ (d) 3

17. $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix} =$

(a) 0 (b) $2abc$
(c) $a^2b^2c^2$ (d) None of these

18. If $\begin{vmatrix} x+1 & 1 & 1 \\ 2 & x+2 & 2 \\ 3 & 3 & x+3 \end{vmatrix} = 0$, then x is

(a) 0, – 6 (b) 0, 6
(c) 6 (d) None of these

19. The roots of the equation $\begin{vmatrix} x & 0 & 8 \\ 4 & 1 & 3 \\ 2 & 0 & x \end{vmatrix} = 0$ are equal to

(a) –4, 4 (b) 2, –4 (c) 2, 4 (d) 2, 8

20. If $\begin{vmatrix} 5 & 3 & -1 \\ -7 & x & -3 \\ 9 & 6 & -2 \end{vmatrix} = 0$, then x is equal to

(a) 3 (b) 5 (c) 7 (d) 9

21. A, B are n-rowed square matrices such that $AB = O$ and B is non-singular. Then

(a) $A \neq O$ (b) $A = O$
(c) $A = I$ (d) None of these

22. The solutions of the equation $\begin{vmatrix} x & 2 & -1 \\ 2 & 5 & x \\ -1 & 2 & x \end{vmatrix} = 0$ are

(a) 3, –1 (b) –3, 1 (c) 3, 1 (d) –3, –1

23. If $f(x)=\begin{vmatrix} x-3 & 2x^2-18 & 3x^3-81 \\ x-5 & 2x^2-50 & 4x^3-500 \\ 1 & 2 & 3 \end{vmatrix}$

then $f(1)\cdot f(3)+f(3)\cdot f(5)+f(5)\cdot f(1)=$

(a) $f(1)$ (b) $f(3)$

(c) $f(1)+f(3)$ (d) $f(1)+f(5)$

24. The value of k for which the set of equations $x+ky+3z=0$, $3x+ky-2z=0$, $2x+3y-4z=0$ has a non-trivial solution is

(a) 15 (b) 31/2

(c) 16 (d) 33/2

25. If the system of equations, $x+2y-3z=1$, $(k+3)z=3$, $(2k+1)x+z=0$ is inconsistent, then the value of k is

(a) -3 (b) 1/2 (c) 0 (d) 2

26. The system of linear equations $x+y+z=2$, $2x+y-z=3$, $3x+2y+kz=4$ has a unique solution if

(a) $k\neq 0$ (b) $-1<k<1$

(c) $-2<k<2$ (d) $k=0$

27. The system of equations $x+y+z=2$, $3x-y+2z=6$ and $3x+y+z=-18$ has

(a) a unique solution

(b) no solutions

(c) an infinite number of solutions

(d) zero solution as the only solution

28. The system of equations

$\alpha x+y+z=\alpha-1$

$x+\alpha y+z=\alpha-1$

$x+y+\alpha z=\alpha-1$

has no solution, if α is

(a) not equal to -2 (b) 1

(c) -2 (d) either -2 or 1

29. If A is a square matrix of order n and $A=kB$, where k is a scalar, then $|A|=$

(a) $|B|$ (b) $k|B|$

(c) $k^n|B|$ (d) $n|B|$

30. If $A=[a\ b]$, $B=[-b\ -a]$ and $C=\begin{bmatrix} a \\ -a \end{bmatrix}$, then the correct statement is

(a) $A=-B$ (b) $A+B=A-B$

(c) $AC=BC$ (d) $CA=CB$

Assertion & Reason Type

Directions : *In the following questions, Statement-1 is followed by Statement-2. Mark the correct choice as :*

(a) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is true, Statement–2 is true, Statement-2 is not a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

31. **Statement-1:** The determinants

$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ and $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ are not identical.

Statement-2: The first two columns in both the determinants are identical and third column is different.

32. **Statement-1 :** If $A=\begin{bmatrix} 2 & 1+2i \\ 1-2i & 7 \end{bmatrix}$, then $\det(A)$ is real.

Statement-2 : If $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, a_{ij} being complex numbers when $i\neq j$, then $\det(A)$ is always real.

33. **Statement-1 :** If A is a matrix of order 2×2, then $|\text{adj}A|=|A|$.

Statement-2 : $|A|=|A^T|$.

34. **Statement-1 :** If $a_1, a_2, \ldots\ldots, a_n, \ldots$ are in G.P. ($a_i>0$ for all i), then $\Delta=\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}=0$

Statement-2 : The three elements in any row of the determinant are in H.P.

35. **Statement-1 :** If $A=\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$, then $|A|=0$

Statement-2 : The value of the determinant of a skew symmetric matrix is always zero.

36. **Statement-1 :** The inverse of the matrix

$A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ does not exist.

Statement-2 : If determinant of matrix is zero, then the inverse of that matrix does not exist.

Comprehension Type

Paragraph for Q. No. 37-39

Let $A=\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix}$. If U_1, U_2 and U_3 are column matrices satisfying

$AU_1=\begin{bmatrix}1\\0\\0\end{bmatrix}, AU_2=\begin{bmatrix}2\\3\\0\end{bmatrix}$ and $AU_3=\begin{bmatrix}2\\3\\1\end{bmatrix}$, U is 3×3 matrix whose columns are U_1, U_2, U_3. Then,

37. The value of $|U|$ is

(a) 3 (b) −3 (c) 3/2 (d) 2

38. The sum of the elements of U^{-1} is

(a) −1 (b) 0 (c) 1 (d) 3

39. The value of $[3\ 2\ 0]\ U\begin{bmatrix}3\\2\\0\end{bmatrix}$ is

(a) [5] (b) $\left[\frac{5}{2}\right]$ (c) [4] (d) $\left[\frac{3}{2}\right]$

Matrix-Match Type

40. Match the following :

	Column-I		Column-II
A.	Let $\lvert A\rvert = \lvert a_{ij}\rvert_{3\times 3} \neq 0$. Each element a_{ij} is multiplied by k^{i-j}. Let $\lvert B\rvert$ the resulting determinant, where $k_1\lvert A\rvert + k_2\lvert B\rvert = 0$. Then $k_1 + k_2 =$	p.	0
B.	The maximum value of a third order determinant each of its entries are ± 1 equals	q.	4
C.	$\begin{vmatrix}1&\cos\alpha&\cos\beta\\\cos\alpha&1&\cos\gamma\\\cos\beta&\cos\gamma&1\end{vmatrix} = \begin{vmatrix}0&\cos\alpha&\cos\beta\\\cos\alpha&0&\cos\gamma\\\cos\beta&\cos\gamma&0\end{vmatrix}$ if $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=$	r.	1
D.	$\begin{vmatrix}x^2+x&x+1&x-2\\2x^2+3x-1&3x&3x-3\\x^2+2x+3&2x-1&2x-1\end{vmatrix} = Ax+B$ then $A+2B$ is	s.	$\begin{vmatrix}1&2\\2&4\end{vmatrix}$

SOLUTIONS

1. (a) : On putting $x = 0$, the determinant (Δ) becomes,

$$\Delta_{x=0}=\begin{vmatrix}0&-a&-b\\a&0&-c\\b&c&0\end{vmatrix}=a(bc)-b(ac)=0$$

$\therefore$ $x = 0$ is a root of the given equation.

2. (d) : We have, $\begin{vmatrix}x+1&3&5\\2&x+2&5\\2&3&x+4\end{vmatrix}=0$

Apply $C_1\to C_1+C_2+C_3$, we get

$$(9+x)\begin{vmatrix}1&3&5\\1&x+2&5\\1&3&x+4\end{vmatrix}=0$$

Apply $R_1\to R_1-R_2$, we get

$$(x+9)\begin{vmatrix}0&1-x&0\\1&x+2&5\\1&3&x+4\end{vmatrix}=0$$

Apply $R_2\to R_2-R_3$, we get

$$\Rightarrow (x+9)\begin{vmatrix}0&1-x&0\\0&-(1-x)&1-x\\1&3&x+4\end{vmatrix}=0$$

$$\Rightarrow (x+9)(1-x)\begin{vmatrix}0&1&0\\0&-1&1-x\\1&3&x+4\end{vmatrix}=0$$

$\Rightarrow (x+9)(1-x)(1-x)=0$

$\Rightarrow x = 1, 1, -9$

3. (c) : $\begin{vmatrix}0&a&-b\\-a&0&c\\b&-c&0\end{vmatrix}=0$

(Since value of determinant of skew-symmetric matrix of odd order is 0).

4. **(b)** : $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x+y+z)\begin{vmatrix} 2 & 1 & 1 \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$

[Using $R_1 \to R_1 + R_2 + R_3$]

$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & z & x \\ x & y & z \end{vmatrix}$; [Using $C_1 \to C_1 - C_2$]

$= (x + y + z)\{(z^2 - xy) - (xz - x^2) + (xy - xz)\}$
$= (x + y + z)(x - z)^2$

Compare with given condition, we get $k = 1$.

5. **(b)** : $\Delta = \begin{vmatrix} 2(a+b+c) & 0 & a+b+c \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$

[Using $R_1 \to R_1 + R_2 + R_3$]

$\Delta = (a+b+c)\begin{vmatrix} 2 & 0 & 1 \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$

On expanding, $\Delta = -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
$= -(a^3 + b^3 + c^3 - 3abc) = 3abc - a^3 - b^3 - c^3$.

6. **(a)** : Let $\Delta \equiv \begin{vmatrix} 2 & 2\omega & -\omega^2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix}$

$= \begin{vmatrix} 2+2\omega+2\omega^2 & 2\omega & -\omega^2 \\ 1+1-2 & 1 & 1 \\ 1-1-0 & -1 & 0 \end{vmatrix}$

[Applying $C_1 \to C_1 + C_2 - 2C_3$].

$= \begin{vmatrix} 0 & 2\omega & -\omega^2 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{vmatrix} = 0$ [$\because 1 + \omega + \omega^2 = 0$]

7. **(a)** : $\begin{vmatrix} 1 & \omega & -\omega^2/2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{vmatrix}$

$= -\frac{1}{2}\begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & 1 & -2 \\ 0 & -1 & 0 \end{vmatrix}$ (Applying $C_1 \to C_1 + C_2 + C_3$) $= 0$

8. **(a)** : $\begin{vmatrix} a_1 & ma_1 & b_1 \\ a_2 & ma_2 & b_2 \\ a_3 & ma_3 & b_3 \end{vmatrix} = m\begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} = 0$

{$\because C_1 \equiv C_2$}

9. **(b)** : Let $\Delta = \begin{vmatrix} 11 & 12 & 13 \\ 12 & 13 & 14 \\ 13 & 14 & 15 \end{vmatrix}$

Apply $C_3 \to C_3 - C_2$, we get

$\Delta = \begin{vmatrix} 11 & 12 & 1 \\ 12 & 13 & 1 \\ 13 & 14 & 1 \end{vmatrix}$

Apply $C_2 \to C_2 - C_1$, we get

$\Delta = \begin{vmatrix} 11 & 1 & 1 \\ 12 & 1 & 1 \\ 13 & 1 & 1 \end{vmatrix} = 0$ [$\because C_2 \equiv C_3$]

10. **(c)** : Applying $C_1 \to C_1 + C_2 + C_3$, we obtain

$-x\begin{vmatrix} 1 & -6 & 3 \\ 1 & 3-x & 3 \\ 1 & 3 & -6-x \end{vmatrix} = 0$

Applying $R_1 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$, we get

$\Rightarrow -x\begin{vmatrix} 1 & -6 & 3 \\ 0 & 9-x & 0 \\ 0 & 9 & -9-x \end{vmatrix} = 0$

$\Rightarrow -x(9 - x)(-9 - x) = 0 \Rightarrow x = 0, 9, -9$.

11. **(b)** : We have, $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$(x+1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$(x+1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & x-2 & 0 \\ 0 & 0 & x-2 \end{vmatrix} = 0$

$\Rightarrow (x + 1)(x - 2)^2 = 0 \Rightarrow x = -1, 2$.

12. **(a)** : Obviously, the determinant is satisfied for $x = a, b$.

13. **(d)** : Let $A = \begin{vmatrix} x+2 & x+3 & x+a \\ x+4 & x+5 & x+b \\ x+6 & x+7 & x+c \end{vmatrix}$

Applying $C_2 \to C_2 - C_1$, we get

$$A=\begin{vmatrix}x+2 & 1 & x+a\\ x+4 & 1 & x+b\\ x+6 & 1 & x+c\end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow A=\begin{vmatrix}x+2 & 1 & x+a\\ 2 & 0 & b-a\\ 4 & 0 & c-a\end{vmatrix}$$

$= -1\,(2c - 2a - 4b + 4a) = 2(2b - c - a)$

Since, a, b, c are in A.P. $\Rightarrow A = 0$.

14. (c) : We have, $\begin{vmatrix}a & 2b & 2c\\ 3 & b & c\\ 4 & a & b\end{vmatrix}=0$

$$\Rightarrow \begin{vmatrix}a-6 & 0 & 0\\ 3 & b & c\\ 4 & a & b\end{vmatrix}=0$$

[Applying $R_1 \to R_1 - 2R_2$]

$\Rightarrow (a-6)(b^2 - ac) = 0 \Rightarrow b^2 - ac = 0$ ($\because a \neq 6$)

$\therefore\ ac = b^2 \Rightarrow abc = b^3$.

15. (b) : $AB=\begin{bmatrix}1 & 0\\ 2 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 12\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$

while $BA=\begin{bmatrix}0 & 0\\ 1 & 12\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 25 & 0\end{bmatrix}\neq O$

16. (a) : $\begin{vmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{vmatrix}=3(\omega-\omega^2)$

$$=3\left[\frac{-1+\sqrt{3}i}{2}-\frac{-1-\sqrt{3}i}{2}\right]=3\sqrt{3}\,i$$

17. (a) : Put $x = 0$, in given determinant, we get

$$\begin{vmatrix}4 & 0 & 1\\ 4 & 0 & 1\\ 4 & 0 & 1\end{vmatrix}=0$$

18. (a)

19. (a) : Let $\Delta = \begin{vmatrix}x & 0 & 8\\ 4 & 1 & 3\\ 2 & 0 & x\end{vmatrix}$

$\therefore\ \Delta = x(x - 0) - 0(4x - 6) + 8(0 - 2)$

or $x^2 - 16 = 0$

$\Rightarrow x = 4, -4$.

20. (d) : Given, $\begin{vmatrix}5 & 3 & -1\\ -7 & x & -3\\ 9 & 6 & -2\end{vmatrix}=0$

$\Rightarrow 5(-2x + 18) - 3(14 + 27) - 1(-42 - 9x) = 0$

$\therefore\ -10x + 90 - 42 - 81 + 42 + 9x = 0 \Rightarrow x = 9$.

21. (b) : Since $|B| \neq 0 \Rightarrow B^{-1}$ exists.

Now, $AB = O$

$\Rightarrow (AB)B^{-1} = OB^{-1} \Rightarrow A(BB^{-1}) = O$

$\Rightarrow AI = O \Rightarrow A = O$

So, AB and BA are defined only.

22. (a) : $\begin{vmatrix}x & 2 & -1\\ 2 & 5 & x\\ -1 & 2 & x\end{vmatrix}=0$

$\Rightarrow x(5x - 2x) - 2(2x + x) - 1(4 + 5) = 0$

$\Rightarrow 3x^2 - 6x - 9 = 0$

or $x^2 - 2x - 3 = 0$

$\Rightarrow (x + 1)(x - 3) = 0$ or $x = -1, 3$.

23. (b) : Taking out $(x - 3)$, $(x - 5)$ and 2 from I row, II row and II column respectively, we get

$$f(x) = 2(x-3)(x-5)\begin{vmatrix}1 & x+3 & 3(x^2+3x+9)\\ 1 & x+5 & 4(x^2+5x+25)\\ 1 & 1 & 3\end{vmatrix}$$

$$\Rightarrow f(x) = 2(x-3)(x-5)\begin{vmatrix}0 & x+2 & 3(x^2+3x+8)\\ 0 & 2 & x^2+11x+73\\ 1 & 1 & 3\end{vmatrix},$$

Applying $R_2 \to R_2 - R_1$

$$=2(x-3)(x-5)\begin{vmatrix}1 & x+3 & 3(x^2+3x+9)\\ 0 & 2 & x^2+11x+73\\ 1 & 1 & 3\end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$, we get

$= 2(x-3)(x-5)[1(x+2)(x^2+11x+73) - 6(x^2+3x+8)]$

$= 2(x^2 - 8x + 15)(x^3 + 13x^2 + 95x + 146 - 6x^2 - 18x - 48)$

$= 2(x^2 - 8x + 15)(x^3 + 7x^2 + 77x + 98)$

$= 2(x^5 - x^4 + 36x^3 - 413x^2 + 371x + 1470)$

$f(1) = 2928, f(3) = 0, f(5) = 0$

$\therefore\ f(1)\cdot f(3) + f(3)\cdot f(5) + f(5)\cdot f(1) = 0 + 0 + 0 = 0 = f(3)$

24. (d) : Given set of equations will have a non-trivial solution, if the determinant of coefficient of x, y, z is zero.

i.e., $\begin{vmatrix}1 & k & 3\\ 3 & k & -2\\ 2 & 3 & -4\end{vmatrix}=0$

$\Rightarrow 2k - 33 = 0 \Rightarrow k = \dfrac{33}{2}$.

25. (a) : For the equation to be inconsistent $D = 0$

$$\therefore\ D = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & k+3 \\ 2k+1 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow\ k = -3$

and $D_1 = \begin{vmatrix} 1 & 2 & -3 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$

$\therefore$ System is inconsistent for $k = -3$.

26. (a) : The given system of equations has a unique solution, if $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0 \Rightarrow k \neq 0$.

27. (a) : Given system of equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -18 \end{bmatrix}$$

On solving the above system, we get the unique solution as $x = -10$, $y = -4$, $z = 16$.

28. (c) : For no solution or infinitely many solutions,

$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

$\Rightarrow \alpha = 1, \alpha = -2$

But for $\alpha = 1$, there are infinitely many solutions and when we put $\alpha = -2$ in given system of equations and add them together, L.H.S $\neq$ R.H.S. *i.e.*, no solution.

29. (c) : $\because A = kB \Rightarrow |A| = k^n|B|$, by fundamental concept.

30. (c) : $AC = [a\ b]\begin{bmatrix} a \\ -a \end{bmatrix} = [a^2 - ab]$

$BC = [-b\ -a]\begin{bmatrix} a \\ -a \end{bmatrix} = [a^2 - ab]$

$\therefore\ AC = BC$.

31. (d) : The first determinant can be shown to be equal to second.

32. (c) : Clearly, $|A| = \begin{vmatrix} 2 & 1+2i \\ 1-2i & 7 \end{vmatrix} = 9$, which is real.

33. (b) : $\because\ |\text{adj } A| = |A|^{n-1} = |A|$ (Here $n = 2$)

34. (c) : Let r be the common ratio of given G.P.

$$\Delta = \begin{vmatrix} \log a_1 + (n-1)\log r & \log a_1 + (n)\log r & \log a_1 + (n+1)\log r \\ \log a_1 + (n+2)\log r & \log a_1 + (n+3)\log r & \log a_1 + (n+4)\log r \\ \log a_1 + (n+5)\log r & \log a_1 + (n+6)\log r & \log a_1 + (n+7)\log r \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$= \begin{vmatrix} \log a_1 + (n-1)\log r & \log a_1 + n\log r & \log a_1 + (n+1)\log r \\ \log a_1 + (n+2)\log r & \log a_1 + (n+3)\log r & \log a_1 + (n+4)\log r \\ 3\log r & 3\log r & 3\log r \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, we get

$$= \begin{vmatrix} \log a_1 + (n-1)\log r & \log a_1 + n\log r & \log a_1 + (n+1)\log r \\ 3\log r & 3\log r & 3\log r \\ 3\log r & 3\log r & 3\log r \end{vmatrix} = 0$$

35. (c) : The value of the determinant of skew - symmetric matrix of odd order is zero.

36. (a) : Since, det $A = 0$, so inverse of the matrix does not exist.

37. (a) : Let $U = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Given, $AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ 2x + y \\ 3x + 2y + z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad ...(i)$$

From (i), we get

$x = 1$ ...(ii), $2x + y = 0$ ...(iii), $(3x + 2y + z)$ ...(iv)

Using (ii), (iii) and (iv), we get $x = 1$, $y = -2$, $z = 1$

$\therefore\ U_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Now, $AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \Rightarrow U_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$

and $AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \Rightarrow U_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$

Hence, $U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix}$

$\therefore\ |U| = 1(3 - 4) - 2(6 + 1) + 2(8 + 1) = -1 - 14 + 18 = 3$

38. (b) : adj $U=\begin{bmatrix}-1 & -2 & 0\\ -7 & -5 & -3\\ 9 & 6 & 3\end{bmatrix}$ and $|U|=3$

$\therefore\ U^{-1}=\dfrac{\text{adj}U}{|U|}=\dfrac{1}{3}\begin{bmatrix}-1 & -2 & 0\\ -7 & -5 & -3\\ 9 & 6 & 3\end{bmatrix}$

$\therefore$ Sum of elements of $U^{-1}=0$

39. (a) :

$$\begin{bmatrix}3 & 2 & 0\end{bmatrix}U\begin{bmatrix}3\\2\\0\end{bmatrix}=\begin{bmatrix}3 & 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ -2 & -1 & -1\\ 1 & -4 & -3\end{bmatrix}\begin{bmatrix}3\\2\\0\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 4 & 4\end{bmatrix}\begin{bmatrix}3\\2\\0\end{bmatrix}=\begin{bmatrix}-3+8\end{bmatrix}=\begin{bmatrix}5\end{bmatrix}$$

40. (a) : (A)-(p, s); (B)-(q); (C)-(r); (D)-(p, s)

(A) $|A|=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}\quad |B|=\begin{vmatrix}a_{11} & k^{-1}a_{12} & k^{-2}a_{13}\\ ka_{21} & a_{22} & k^{-1}a_{23}\\ k^2a_{31} & ka_{32} & a_{33}\end{vmatrix}$

$$=\frac{1}{k^3}\begin{vmatrix}k^2a_{11} & ka_{12} & a_{13}\\ k^2a_{21} & ka_{22} & a_{23}\\ k^2a_{31} & ka_{32} & a_{33}\end{vmatrix}=|A|$$

$k_1|A|+k_2|B|=0 \Rightarrow k_1+k_2=0$

(B) $\begin{vmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 1 & 1\end{vmatrix}=4$

(C) $\begin{vmatrix}1 & \cos\alpha & \cos\beta\\ \cos\alpha & 1 & \cos\gamma\\ \cos\beta & \cos\gamma & 1\end{vmatrix}=\begin{vmatrix}0 & \cos\alpha & \cos\beta\\ \cos\alpha & 0 & \cos\gamma\\ \cos\beta & \cos\gamma & 0\end{vmatrix}$

$\Rightarrow\ \sin^2\gamma-\cos\alpha(\cos\alpha-\cos\beta\cos\gamma)$

$\qquad+\cos\beta(\cos\alpha\cos\gamma-\cos\beta)$

$=-\cos\alpha(-\cos\beta\cos\gamma)+\cos\beta(\cos\alpha\cos\gamma)$

$\Rightarrow\ \sin^2\gamma-\cos^2\alpha+2\cos\alpha\cos\beta\cos\gamma-\cos^2\beta$

$=2\cos\alpha\cos\beta\cos\gamma\ \Rightarrow\ \sin^2\gamma=\cos^2\alpha+\cos^2\beta$

$\Rightarrow\ \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

(D) Let $\Delta=\begin{vmatrix}x^2+x & x+1 & x-2\\ 2x^2+3x-1 & 3x & 3x-3\\ x^2+2x+3 & 2x-1 & 2x-1\end{vmatrix}$

Applying $R_2\to R_2-(R_1+R_3)$, we get

$$\Delta=\begin{vmatrix}x^2+x & x+1 & x-2\\ -4 & 0 & 0\\ x^2+2x+3 & 2x-1 & 2x-1\end{vmatrix}=4\begin{vmatrix}x+1 & x-2\\ 2x-1 & 2x-1\end{vmatrix}$$

$$=4\begin{vmatrix}x+1 & -3\\ 2x-1 & 0\end{vmatrix}=24x-12$$

$\therefore\ A=24,\ B=-12$ So, $A+2B=0$

◈◈

CBSE warm-up!

CLASS-XII

Chapterwise Practice questions for CBSE Exams as per the latest pattern and marking scheme issued by CBSE for the academic session 2019-20.

Series 1 — Relations and Functions

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(i) All questions are compulsory.

(ii) This question paper contains **36** questions.

(iii) Question **1-20** in Section-A are very short-answer-objective type questions carrying **1** mark each.

(iv) Question **21-26** in Section-B are short-answer type questions carrying **2** marks each.

(v) Question **27-32** in Section-C are long-answer-**I** type questions carrying **4** marks each.

(vi) Question **33-36** in Section-D are long-answer-**II** type questions carrying **6** marks each.

SECTION - A

(Q.1 - Q.10) are multiple choice type questions. Select the correct option.

1. The relations $R = \{(1, 1), (3, 3), (4, 4)\}$ on the set $\{1, 3, 4\}$ is

(a) symmetric only (b) reflexive only
(c) an equivalence relation
(d) transitive only

2. A mapping $f: A \to B$ is one-one, if

(a) $f(a_1) \neq f(a_2)$ for all $a_1, a_2 \in$ A
(b) $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ for all $a_1, a_2 \in A$
(c) $a_1 = a_2 \Rightarrow f(a_1) = f(a_2)$ for all $a_1, a_2 \in A$
(d) none of these

3. Let $R = \{(2, 2), (4, 4), (6, 6), (8, 8), (2, 4), (4, 6), (2, 6)\}$ be a relation on the set $X = \{2, 4, 6, 8\}$, then relation R is

(a) reflexive and symmetric only
(b) an equivalence relation
(c) reflexive only
(d) reflexive and transitive only

4. $f: R \to R$ given by $f(x) = n + \sqrt{x^2}$, is

(a) injective (b) surjective
(c) bijective (d) none of these

5. If $f : R \to R$ and $g : R \to R$ defined by $f(x) = 3x - 4$ and $g(x) = x^2 + 3$, then the value of x for which $f(g(x)) = 32$ is

(a) ± 2 (b) ± 3 (c) ± 1 (d) ± 4

6. Let $R = \{(1, 3), (4, 2), (2, 4), (3, 2), (3, 1)\}$ be a relation on the set $X = \{1, 2, 3, 4\}$. The relation R is

(a) a function (b) transitive
(c) not symmetric (d) reflexive

7. If $f: R \to R$ is given by $f(x) = 2x - 7$, then $f^{-1}(x) =$

(a) $\frac{1}{2x-7}$ (b) $\frac{x+7}{2}$
(c) $\frac{1}{2x+7}$ (d) none of these.

8. Let $f(x) = x^2$ and $g(x) = 2^x$, then $fog(x) =$

(a) 2^{2x} (b) x^{2x}
(c) 2^{x^2} (d) none of these

9. Two functions $f: R \to R$ and $g: R \to R$ are defined as follows:

$$f(x)=\begin{cases}1, & x \notin Q \\ 0, & x \in Q\end{cases} \qquad g(x)=\begin{cases}0, & x \notin Q \\ -1, & x \in Q\end{cases}$$

Then, $gof(e) + fog(\pi) =$

(a) 0 (b) 1 (c) –1 (d) 2

10. If $f(x) = \cos^2 x$ and the composite function $g(f(x)) = |\cos x|$, then $g(x)$ is equal to

(a) $\sqrt{x-1}$ (b) $\sqrt{x}$

(c) $\sqrt{x+1}$ (d) $-\sqrt{x}$

(Q. 11-Q.15) Fill in the blanks.

11. Let $R = \{(a, a^3) : a$ is a prime number less than 5} be a relation, then the range of R is ______.

12. If $f : R \to R$ is defined by $f(x) = 3x + 2$, then $f(f(x)) =$ ______.

13. If the function $f: R \to R$, defined by $f(x) = 3x - 4$, is invertible, then $f^{-1} =$ ______.

14. Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$. Ordered pair ______ needs to be added to R to make it the smallest equivalence relation.

OR

If $A = \{1, 2, 3, 4\}$ and $f = \{(1, 2), (2, 4), (3, 1), (4, 3)\}$ then $f^{-1} =$ ______

15. Let D be the domain of the real valued function f defined by $f(x) = \sqrt{49-x^2}$. Then D is given by ______.

OR

If functions $f: A \to B$ and $g: B \to A$ satisfy $gof = I_A$, then f and g are ______ functions.

(Q. 16-Q.20) Answer the following questions.

16. Let R be the set of all real numbers. Let $f: R \to R$ such that $f(x) = \sin x$ and $g : R \to R$ such that $g(x) = x^2$. Prove that $gof \neq fog$.

17. Let N be the set of natural numbers and relation R on N be defined by $R = \{(x, y); x, y \in N, x + 4y = 10\}$ Determine whether the above relation is reflexive, symmetric.

18. Let $f: R \to R$ be defined as $f(x) = x^2 + 1$. Find :

(i) $f^{-1}(-5)$ (ii) $f^{-1}\{10, 37\}$

OR

Let $A = \{3, 4, 5\}$ and relation R on set A is defined as $R = \{(a, b) \in A \times A : a - b = 10\}$. Is relation an empty relation?

19. Test the function for one-one and onto :
$f: R \to R$ defined by $f(x) = x^2 + 3$; $\forall x \in R$

20. Let $f: R \to R$ and $g : R \to R$ be defined by $f(x) = x^2$, $g(x) = x + 2$; $\forall x \in R$ (set of all real numbers). Is $gof = fog$?

SECTION-B

21. If $f(x)$ is an invertible function, then find the inverse of $f(x)=\frac{3x-2}{5}$.

OR

Let $f : Z \to Z$ be defined by $f(x) = x + 2$. Find $g : Z \to Z$ such that $gof = I_Z$.

22. Let $f: R - \left\{-\frac{4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ be a function defined as $f(x)=\frac{4x}{3x+4}$, find f^{-1} : Range of $f \to R - \left\{-\frac{4}{3}\right\}$.

23. Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

24. If $f, g : R \to R$ are defined respectively by $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$, then find

(i) fof (ii) gog

OR

Is $f: R \to R$ defined by $f(x) = |x| + x$ is one-one or onto? Also find the range of f.

25. If $f: R \to R$ is defined by $f(x) = (3 - x^3)^{1/3}$, then find $fof(x)$.

26. Show that the function $f : N \to N$ defined by $f(x) = x^3$ is injective but not surjective.

SECTION-C

27. Let R be a relation defined on the set of natural numbers N as follow :
$R = \{(x, y) \mid x \in N, y \in N$ and $2x + y = 24\}$
Find the domain and range of the relation R. Also, find if R is an equivalence relation or not.

28. Show that the function $f : R \to R$ given by $f(x) = ax + b$, where $a, b \in R$, $a \neq 0$ is a bijective function.

OR

Let $A = R - \left\{\frac{3}{5}\right\}$ and $B = R - \left\{\frac{7}{5}\right\}$.

Let $f : A \to B$; $f(x)=\frac{7x+4}{5x-3}$ and $g : B \to A$;

$g(y) = \frac{3y+4}{5y-7}$. Show that $(gof) = I_A$ and $(fog) = I_B$.

29. If Q is the set of rational numbers and R is a relation defined on Q by $xRy \Leftrightarrow |x-y| \le \frac{1}{2}$, then prove that R is not an equivalence relation.

30. If $f : R \to R$ be the function defined by $f(x) = 4x^3 + 7$, show that f is a bijection.

31. Let $f : N \to Y$, where $f(x) = 4x^2 + 12x + 15$ and Y = range (f). Show that f is invertible and find f^{-1}.

OR

Show that the relation S in the set $A = \{x \in Z : 0 \le x \le 12\}$ given by $S = \{(a, b) : a, b \in Z, |a - b|$ is divisible by 4$\}$ is an equivalence relation. Find the set of all elements related to 1.

32. Show that the function f in $A = R - \left\{\frac{2}{3}\right\}$ defined as $f(x) = \frac{4x+3}{6x-4}$ is one-one and onto. Hence find f^{-1}.

SECTION - D

33. Let $f(x) = [x]$ and $g(x) = |x|$. Find

(i) $(gof)\left(\frac{-5}{3}\right) - (fog)\left(\frac{-5}{3}\right)$

(ii) $(gof)\left(\frac{5}{3}\right) - (fog)\left(\frac{5}{3}\right)$

(iii) $(f + 2g)(-1)$

OR

Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by $(a, b)\ R(c, d)$ if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation.

34. Prove that the intersection of two equivalence relations is also an equivalence relation.

35. Let P be the set of all the points in a plane and the relation R in set P be defined as $R = \{(A, B) \in P \times P|$ distance between points A and B is less than 3 units$\}$. Show that the relation R is not an equivalence relation.

OR

Consider $f: R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{\sqrt{y+6}-1}{3}\right)$.

Hence find

(i) $f^{-1}(10)$

(ii) y if $f^{-1}(y) = \frac{4}{3}$

where R_+ is the set of all non-negative real numbers.

36. Let $f: W \to W : f(n) = \begin{cases}(n-1), \text{ when } n \text{ is odd}\\(n+1), \text{ when } n \text{ is even.}\end{cases}$

Show that f is invertible. Find f^{-1}.

SOLUTIONS

1. (c) : Clearly, R is the identity relation on set $\{1, 3, 4\}$ and the identity relation is always an equivalence relation. Hence, R is an equivalence relation.

2. (b)

3. (d) : Clearly R is reflexive on X.
R is not symmetric because $(2, 4) \in R$ but $(4, 2) \notin R$.
R is transitive on X also.

4. (d) : We have, $f(x) = x + \sqrt{x^2} = x + |x|$

$$= \begin{cases} 2x, & x \ge 0 \\ x - x = 0, & x < 0 \end{cases}$$

Clearly, f is many - one into function.

5. (b) : We have, $f(g(x)) = 32$
$\Rightarrow f(x^2 + 3) = 32 \Rightarrow 3(x^2 + 3) - 4 = 32$
$\Rightarrow 3x^2 + 9 - 4 = 32 \Rightarrow 3x^2 = 27 \Rightarrow x^2 = 9$
$\Rightarrow x = \pm 3$

6. (c) : Clearly $(3, 2) \in R$ but $(2, 3) \notin R$.
So, R is not symmetric.

7. (b) : Let $y = f(x) = 2x - 7$
$\Rightarrow 2x = y + 7 \Rightarrow x = \frac{y+7}{2}$
Hence, $f^{-1}(x) = \frac{x+7}{2}$

8. (a) : We have, $f(x) = x^2$ and $g(x) = 2^x$
$\therefore fog(x) = f(g(x)) = f(2^x) = (2^x)^2 = 2^{2x}$

9. (c) : We have, $gof(e) + fog(\pi)$
$= g(f(e)) + f(g(\pi))$
$= g(1) + f(0) = -1 + 0 = -1$

10. (b) : We have, $f(x) = \cos^2 x$ and $g(f(x)) = |\cos x|$
Now, $g(f(x)) = |\cos x| = \sqrt{\cos^2 x}$
$\Rightarrow g(\cos^2 x) = \sqrt{\cos^2 x}$
$\therefore g(x) = \sqrt{x}$

11. Given relation is
$R = \{(a, a^3) : a$ is a prime number less than 5$\}$.
$\therefore R = \{(2, 8), (3, 27)\}$
So, the range of R is $\{8, 27\}$.

12. We have, $f(x) = 3x + 2$

$\therefore\ f(f(x)) = f(3x + 2) = 3(3x + 2) + 2 = 9x + 8$

13. We have, $f(x) = 3x - 4$

Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$\therefore\ y = 3x - 4 \Rightarrow x = \frac{y+4}{3}$

$\Rightarrow\ f^{-1}(y) = \frac{y+4}{3} \Rightarrow f^{-1}(x) = \frac{x+4}{3}$

14. (3, 1) needs to be added to R to make it the smallest equivalence relation.

OR

$f^{-1} = \{(2, 1), (4, 2), (1, 3), (3, 4)\}$

15. We have, $f(x) = \sqrt{49 - x^2}$

It is defined if $49 - x^2 \geq 0$

$\Rightarrow\ x^2 \leq 49 \Rightarrow -7 \leq x \leq 7$

Hence domain D is $[-7, 7]$.

OR

Given that $f : A \to B$ and $g : B \to A$ satisfy $gof = I_A$. Clearly function g is inverse off. Therefore f has to be bijective. So, g is also bijective.

16. Let x be an arbitrary real number, then

$(gof)(x) = g\{f(x)\} = g(\sin x) = (\sin x)^2$

$(fog)(x) = f\{g(x)\} = f(x^2) = \sin x^2$

Clearly, $(\sin x)^2 \neq \sin x^2$.

Hence, $gof \neq fog$.

17. Given, $R = \{(x, y); x, y \in N, x + 4y = 10\}$

$\therefore\ R = \{(2, 2), (6, 1)\}$

R is not reflexive because $(1, 1) \notin R$.

R is not symmetric, because $(6, 1) \in R$ but $(1, 6) \notin R$.

18. (i) Let $f^{-1}(-5) = x$. Then,

$f(x) = -5 \Rightarrow x^2 + 1 = -5 \Rightarrow x^2 = -6 \Rightarrow x = \pm\sqrt{-6}$, which is not in R.

So, $f^{-1}(-5) = \phi$.

(ii) $f^{-1}\{10, 37\} = \{x \in R : f(x) = 10 \text{ or } f(x) = 37\}$

$= \{x \in R : x^2 + 1 = 10 \text{ or } x^2 + 1 = 37\}$

$= \{x \in R : x^2 = 9 \text{ or } x^2 = 36\} = \{3, -3, 6, -6\}$

OR

We notice for no value of $a, b \in A$, $a - b = 10$. Hence, $(a, b) \notin R$ for $a, b \in A$. Hence, R is an empty relation.

19. Given, $f : R \to R$ defined by $f(x) = x^2 + 3\ \forall\ x \in R$

Clearly $-1, 1 \in R$ and $f(-1) = 4$ and $f(1) = 4$

Thus two different elements -1 and 1 of domain have same image under f and hence f is not one-one.

Since, $f(x) = x^2 + 3 \geq 3$ $\quad [\because x^2 \geq 0]$

Hence, range $f = [3, \infty) = \{y : 3 \leq y < \infty\} \neq$ codomain R

Hence, f is not onto

Thus f is neither one-one nor onto.

20. $(gof)(x) = g[f(x)] = g(x^2) = x^2 + 2$

$(fog)(x) = f[g(x)] = f(x + 2) = (x + 2)^2$

Hence, $gof \neq fog$.

21. Let $y = f(x) = \frac{3x-2}{5}$, then $D_f = R$ and $R_f = R$

$\Rightarrow\ 5y = 3x - 2\ \Rightarrow\ 5y + 2 = 3x$

$\Rightarrow\ x = \frac{5y+2}{3}, \forall\ x, y \in R \Rightarrow f^{-1}(y) = \frac{5y+2}{3}, \forall, y \in R$

$\therefore\ f^{-1}(x) = \frac{5x+2}{3}$

OR

Given $f(x) = x + 2$

Now, $gof = I_Z\ \Rightarrow (gof)(x) = I_Z(x)$, for all $x \in Z$

$\Rightarrow\ g(f(x)) = x$, for all $x \in Z$

$\Rightarrow\ g(x + 2) = x$, for all $x \in Z$

$\Rightarrow\ g(x) = x - 2$, for all $x \in Z$ $\quad$ [Replace x by $x - 2$]

Hence the required function $g : Z \to Z$ is given by $g(x) = x - 2$, for all $x \in Z$.

22. Let $y = f(x)$, Then, $y = \frac{4x}{3x+4} \Rightarrow 3xy + 4y = 4x$

$\Rightarrow\ x(4 - 3y) = 4y$

$\Rightarrow\ x = \frac{4y}{4-3y}$

$\therefore\ f^{-1}(y) = \frac{4y}{4-3y}$ or $f^{-1}(x) = \frac{4x}{4-3x}$

23. For one-one : Let $x_1, x_2 \in R_+$ and consider $f(x_1) = f(x_2)$

$\Rightarrow x_1^2 + 4 = x_2^2 + 4$

$\Rightarrow\ x_1^2 = x_2^2 \Rightarrow x_1 = x_2\quad (\because x_1, x_2 \in R_+)$. So, f is one-one.

For onto : Let $y \in [4, \infty)$ and $x \in R_+$ such that $f(x) = y$

$\Rightarrow\ x^2 + 4 = y \Rightarrow x = \sqrt{y-4} \in R_+$. Hence, f is onto.

Since, f is one-one and onto.

$\therefore\ f$ is invertible, with $x = \sqrt{y-4}$ or $f^{-1}(y) = \sqrt{y-4}$.

24. Given, $f(x) = x^2 + 3x + 1$ and $g(x) = 2x - 3$

(i) For any $x \in R$, we have

$(fof)(x) = f(f(x)) = f(x^2 + 3x + 1)$

$= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1$

$= x^4 + 9x^2 + 1^2 + 6x^3 + 2x^2 + 6x + 3(x^2 + 3x + 1) + 1$

$= x^4 + 6x^3 + 14x^2 + 15x + 5$

Hence, $fof : R \to R$ is defined by

$(fof)(x) = x^4 + 6x^3 + 14x^2 + 15x + 5$, for all $x \in R$

(ii) For any $x \in R$, we have

$(gog)(x) = g(g(x)) = g(2x - 3) = 2(2x - 3) - 3 = 4x - 9$

Hence, $gog : R \to R$ is defined by $(gog)(x) = 4x - 9$, for all $x \in R$

OR

When $x \geq 0, |x| = x \Rightarrow f(x) = x + x = 2x$;
when $x < 0, |x| = -x \Rightarrow f(x) = -x + x = 0$
Since, $f(x) = 0$ for all $x < 0 \Rightarrow f(-1) = 0$ and $f(-2) = 0$
So, the two different elements $-1, -2 \in R$ (domain of f) have same image.
$\Rightarrow$ f is not one-one.
When $x \geq 0, f(x) = 2x \geq 0$ and when $x < 0, f(x) = 0$
$\Rightarrow$ Range of $f = [0, \infty)$.
Since, Range of $f = [0, \infty)$, which is a proper subset of R (codomain of f)
$\Rightarrow$ f is not onto.

25. $f : R \to R$ and $f(x) = (3 - x^3)^{1/3}$
$\therefore$ $fof(x) = f(f(x)) = f[(3 - x^3)^{1/3}]$
$= [3 - \{(3 - x^3)^{1/3}\}^3]^{1/3}$
$= [3 - (3 - x^3)]^{1/3} = (3 - 3 + x^3)^{1/3} = x$

26. Let $x_1, x_2 \in N$ be such that $f(x_1) = f(x_2)$
$\Rightarrow x_1^3 = x_2^3 \Rightarrow x_1^3 - x_2^3 = 0$
$\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$
$\Rightarrow x_1 - x_2 = 0$ $\quad (\because x_1, x_2 \in N$, so $x_1^2 + x_1x_2 + x_2^2 > 0)$
$\Rightarrow x_1 = x_2$.
Thus, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \Rightarrow f$ is one-one *i.e.*, injective.
Since, $2 \in N$ (codomain of f) and there does not exist any $x \in N$ (domain of f) such that $f(x) = 2$ *i.e.*, $x^3 = 2$. So, f is not onto *i.e.*, f is not surjective.

27. Here, $R = \{(x, y) \mid x \in N, y \in N$ and $2x + y = 24\}$
Domain of $R = \{1, 2, 3, 4, ..., 11\}$
Range of $R = \{2, 4, 6, 8, 10, 12, ..., 22\}$
R is not reflexive as if $(2, 2) \in R$
$\Rightarrow 2 \times 2 + 2 = 6 \neq 24$
In fact R is neither symmetric nor transitive.
$\Rightarrow$ R is not an equivalence relation.

28. We have $f(x) = ax + b$ where $a, b \in R$ and $a \neq 0$
(i) Injectivity : Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$
$\Rightarrow ax_1 + b = ax_2 + b \Rightarrow x_1 = x_2$
$\therefore$ $f(x)$ is one-one.
(ii) Surjectivity : Let $y \in R$ (co-domain) such that $f(x) = y$
$$\Rightarrow y = ax + b \Rightarrow x = \frac{y-b}{a} \in R \quad (\because a \neq 0)$$
$$\therefore \quad f\left(\frac{y-b}{a}\right) = a\left(\frac{y-b}{a}\right) + b = y$$
$\therefore$ $f(x)$ is onto.
Hence, f is injective and surjective. So, $f(x)$ is bijective.

OR

Let $x \in A$.
Then, $(gof)(x) = g[f(x)]$
$$= g\left(\frac{7x+4}{5x-3}\right) = \frac{3\left(\frac{7x+4}{5x-3}\right)+4}{5\left(\frac{7x+4}{5x-3}\right)-7}$$
$$= \frac{21x+12+20x-12}{35x+20-35x+21} = \frac{41x}{41} = x = I_A(x)$$
$\therefore$ $(gof) = I_A$
Again, let $y \in B$. Then, $(fog)(y) = f[g(y)]$
$$= f\left(\frac{3y+4}{5y-7}\right) = \frac{7\left(\frac{3y+4}{5y-7}\right)+4}{5\left(\frac{3y+4}{5y-7}\right)-3}$$
$$= \frac{21y+28+20y-28}{15y+20-15y+21} = \frac{41y}{41} = y = I_B(y)$$
$\therefore$ $(fog) = I_B$
Hence, $(gof) = I_A$ and $(fog) = I_B$.

29. (i) $|x - x| = 0 \leq \frac{1}{2}; \forall x \in Q$.
$\therefore$ xRx *i.e.*, R is reflexive on Q.
(ii) $xRy \Rightarrow |x - y| \leq \frac{1}{2}$
$\Rightarrow |y - x| \leq \frac{1}{2}$ $\quad [\because |y - x| = |x - y|]$
$\Rightarrow$ yRx *i.e.* R is symmetric on Q.
(iii) Let $x = \frac{1}{4}, y = \frac{3}{5}, z = 1$. Then
$$|x - y| = \left|\frac{1}{4} - \frac{3}{5}\right| = \frac{7}{20} \leq \frac{1}{2}$$
$$|y - z| = \left|\frac{3}{5} - 1\right| = \frac{2}{5} \leq \frac{1}{2}$$
$$|x - z| = \left|\frac{1}{4} - 1\right| = \frac{3}{4} \not\leq \frac{1}{2}$$
$\therefore$ $\frac{1}{4} R \frac{3}{5}$ and $\frac{3}{5} R 1$ but $\frac{1}{4}$ is not R-related to 1.
So R is not transitive. Thus R is not an equivalence relation on Q.

30. *Refer to answer 21, Page No. 8 MTG CBSE Champion Mathematics Class-12*

Monthly Test Drive-1 CLASS XII ANSWER KEY

1.	(b)	**2.**	(d)	**3.**	(d)	**4.**	(b)
5.	(c)	**6.**	(a)	**7.**	(a, b, c)	**8.**	(d)
9.	(a)	**10.**	(b, c, d)	**11.**	(a)	**12.**	(a)
13.	(d)	**14.**	(b)	**15.**	(a)	**16.**	(a)
17.	(1)	**18.**	(5)	**19.**	(2)	**20.**	(3)

31. Let $x_1, x_2 \in N$ such that

$f(x_1) = f(x_2) \Rightarrow 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$

$\Rightarrow 4(x_1^2 - x_2^2) + 12(x_1 - x_2) = 0$

$\Rightarrow (x_1^2 - x_2^2) + 3(x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 3) = 0$

$\Rightarrow x_1 - x_2 = 0 \quad [x_1, x_2 \in N \therefore x_1 + x_2 + 3 \neq 0]$

$\Rightarrow x_1 = x_2 \quad \therefore f$ is one-one

Also, range $(f) = Y$. So, f is onto

Thus, f is one-one onto and therefore invertible.

Let $y \in Y$. Then, for f being onto, there exists $x \in N$ such that $y = f(x)$

Now, $y = f(x) \Rightarrow y = 4x^2 + 12x + 15$

$\Rightarrow y = (2x+3)^2 + 6 \Rightarrow (2x+3) = \sqrt{y-6}$

$\Rightarrow x = \frac{1}{2}\left(\sqrt{y-6} - 3\right)$

$\Rightarrow f^{-1}(y) = \frac{1}{2}\left(\sqrt{y-6} - 3\right)$

Thus, we define $f^{-1} : Y \to N$ as

$f^{-1}(y) = \frac{1}{2}\left(\sqrt{y-6} - 3\right)$

OR

Refer to answer 11, Page No. 7 MTG CBSE Champion Mathematics Class-12

32. Here, $f(x) = \frac{4x+3}{6x-4}$ where $x \in A = R - \left\{\frac{2}{3}\right\}$.

(i) Let $f(x_1) = f(x_2)$ $(\forall\, x_1, x_2 \in A)$

$\Rightarrow \frac{4x_1+3}{6x_1-4} = \frac{4x_2+3}{6x_2-4}$

$\Rightarrow (4x_1 + 3)(6x_2 - 4) = (6x_1 - 4)(4x_2 + 3)$

$\Rightarrow 24x_1x_2 - 16x_1 + 18x_2 - 12 = 24x_1x_2 + 18x_1 - 16x_2 - 12$

$\Rightarrow -34x_1 = -34x_2 \Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

(ii) For $y \in A = R - \left\{\frac{2}{3}\right\}$.

Let $f(x) = y$

$\Leftrightarrow \frac{4x+3}{6x-4} = y \Leftrightarrow (6x-4)y = 4x+3$

$\Leftrightarrow x = \frac{4y+3}{6y-4} \in A\left(\text{as } y \neq \frac{2}{3}\right)$

$\Leftrightarrow f$ is onto and $f(x) = y \Leftrightarrow x = f^{-1}(y)$

$\Leftrightarrow f^{-1}(y) = \frac{4y+3}{6y-4} \forall y \in A$

$\Leftrightarrow f^{-1}(x) = \frac{4x+3}{6x-4} \forall x \in A$

This gives the inverse function of f.

33. We have, $f(x) = [x]$ and $g(x) = |x|$

Clearly, Domain $(f) = R$ and, Domain $(g) = R$. Therefore, each of fog, gof and $f + 2g$ has domain R.

(i) $(gof)\left(\frac{-5}{3}\right) - (fog)\left(\frac{-5}{3}\right) = g\left\{f\left(\frac{-5}{3}\right)\right\} - f\left\{g\left(\frac{-5}{3}\right)\right\}$

$= g\left\{\left[\frac{-5}{3}\right]\right\} - f\left\{\left|\frac{-5}{3}\right|\right\}$

$= g(-2) - f\left(\frac{5}{3}\right) = |-2| - \left[\frac{5}{3}\right] = 2 - 1 = 1$

(ii) $(gof)\left(\frac{5}{3}\right) - (fog)\left(\frac{5}{3}\right) = g\left\{f\left(\frac{5}{3}\right)\right\} - f\left\{g\left(\frac{5}{3}\right)\right\}$

$= g\left\{\left[\frac{5}{3}\right]\right\} - f\left\{\left|\frac{5}{3}\right|\right\} = g(1) - f\left(\frac{5}{3}\right)$

$= |1| - \left[\frac{5}{3}\right] = 1 - 1 = 0$

(iii) $(f + 2g)(-1) = f(-1) + 2g(-1) = [-1] + 2|-1|$
$= -1 + 2 \times 1 = 1.$

OR

Refer to answer 15, Page No. 7 MTG CBSE Champion Mathematics Class-12

34. Let R and S be two equivalence relations on a set A. Then both are reflexive, symmetric and transitive.

(i) $(x, x) \in R$ and $(x, x) \in S$, $\forall\, x \in A$

$\Rightarrow (x, x) \in R \cap S \;\forall\, x \in A$

$\Rightarrow R \cap S$ is reflexive.

(ii) Let $(x, y) \in R \cap S \Rightarrow (x, y) \in R$ and $(x, y) \in S$

$\Rightarrow (y, x) \in R$ and $(y, x) \in S \quad [\because R, S$ are symmetric$]$

$\Rightarrow (y, x) \in R \cap S$

Thus $R \cap S$ is symmetric.

(iii) Let $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$

$\Rightarrow [(x, y) \in R$ and $(x, y) \in S]$ and $[(y, z) \in R$ and $(y, z) \in S]$

$\Rightarrow [(x, y) \in R$ and $(y, z) \in R]$ and $[(x, y) \in S$ and $(y, z) \in S]$

$\Rightarrow (x, z) \in R$ and $(x, z) \in S$ $[\because R$ and S are transitive$]$

$\Rightarrow (x, z) \in R \cap S$.

Hence $R \cap S$ is transitive.

Thus $R \cap S$ is reflexive, symmetric and transitive and therefore it is an equivalence relation on A.

35. Given, $R = \{(A, B) \in P \times P \mid$ distance between points A and B is less than 3 units$\}$

For reflexivity : $(A, A) \in R$ is true as distance between points A and A is 0, which is less than 3 units for all $A \in P$. Hence, R is reflexive. For symmetry : Let $A, B \in P$ $(A, B) \in R \Rightarrow$ distance between points A and B is less than 3 units.

$\Rightarrow$ Distance between B and A is less than 3 units.
So, $(B, A) \in R$
Hence, R is symmetric.
For transitivity : Let points A, B and C are collinear. B is mid-point of AC such that distance between A and B is 2 units and between B and C is also 2 units, *i.e.*, $(A, B) \in R$ and $(B, C) \in R$, we notice distance between A and C is 4 units $\Rightarrow (A, C) \notin R$. Hence, R is not transitive.
Hence, R is not an equivalence relation.

OR

Refer to answer 40, Page No. 11 MTG CBSE Champion Mathematics Class-12

36. Let $f(n_1) = f(n_2)$.
Case 1 : When n_1 is odd and n_2 is even, then
$f(n_1) = f(n_2)$
$\Rightarrow n_1 - 1 = n_2 + 1$
$\Rightarrow n_1 - n_2 = 2$
If n_1 is odd and n_2 is even, then $(n_1 - n_2) \neq 2$.
Thus, we arrive at a contradiction.
So, in this case, $f(n_1) \neq f(n_2)$.
Similarly, when n_1 is even and n_2 is odd, then $f(n_1) \neq f(n_2)$.
Case 2 : When n_1 and n_2 both are odd, then
$f(n_1) = f(n_2)$
$\Rightarrow n_1 - 1 = n_2 - 1$
$\Rightarrow n_1 = n_2$.
Case 3 : When n_1 and n_2 both are even, then
$f(n_1) = f(n_2)$
$\Rightarrow n_1 + 1 = n_2 + 1$
$\Rightarrow n_1 = n_2$.
Thus, from all the cases, we get $f(n_1) = f(n_2) \Rightarrow n_1 = n_2$
$\therefore$ f is one-one.
Now, we show that f is onto.
Let $n \in W$.
Case 1 : When n is odd
In this case, $(n - 1)$ is even
and $f(n - 1) = (n - 1) + 1 = n$. ...(i)
Case 2 : When n is even
In this case, $(n + 1)$ is odd
and $f(n + 1) = (n + 1) - 1 = n$. ...(ii)
Thus, each $n \in W$ has its pre-image in W.
$\therefore$ f is onto.
Thus, f is one-one onto and hence invertible.
Clearly, we have

$$f^{-1}(n) = \begin{cases} (n-1), \text{ when } n \text{ is odd} \\ (n+1), \text{ when } n \text{ is even.} \end{cases} \quad \text{[using (i) and (ii)]}$$

MONTHLY TEST DRIVE

XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80

Series 1 : Relations & Functions

Time Taken : 60 Min.

Only One Option Correct Type

1. Let E denote the words in the English dictionary. Define the relation $R = \{(x, y) \in E \times E$: the word x and y have at least one letter is common$\}$. Then R is

(a) not reflexive, symmetric and transitive.
(b) reflexive, symmetric and not transitive.
(c) reflexive, not symmetric and transitive.
(d) reflexive, symmetric and transitive

2. Which of the following relations is not symmetric?

(a) R_1 on R defined by $(x, y) \in R_1 \Leftrightarrow 1 + xy > 0$ for all $x, y \in R$.
(b) R_2 on $N \times N$ defined by $(a, b)\ R_2\ (c, d) \Leftrightarrow a + d = b + c$ for all $a, b, c, d \in N$.
(c) R_3 on Z defined by $(a, b) \in R_3 \Leftrightarrow b - a$ is an even integer.
(d) R_4 on power set of a set X defined by AR_4B iff $A \subseteq B$.

3. The function $f : R \to R$ defined by $f(x) = 2^x + 2^{|x|}$, is

(a) one-one and onto (b) many-one and onto
(c) one-one and into (d) many-one and into

4. Let $f : [4, \infty) \to [4, \infty)$ be defined by $f(x) = 5^{x(x-4)}$. Then $f^{-1}(x)$ is

(a) $2-\sqrt{4+\log_5 x}$ (b) $2+\sqrt{4+\log_5 x}$
(c) $\left(\frac{1}{5}\right)^{x(x-4)}$ (d) not defined

5. Let X and Y be two non-empty sets and $f : X \to Y$ be a function such that $f(C) = \{f(x) : x \in C\}$ for $C \subseteq X$ and $f^{-1}(D) = \{x : f(x) \in D\}$ for $D \subseteq Y$. If $A \subseteq Y$ and $B \subseteq Y$, then

(a) $f^{-1}(f(A)) = A$
(b) $f^{-1}(f(A)) = A$ only if $f(X) = Y$
(c) $f(f^{-1}(B)) = B$ only if $B \subseteq f(X)$
(d) $f(f^{-1}(B)) = B$

6. The domain of the function $f(x)=\frac{1}{\sqrt{|x|-x}}$ is

(a) $(-\infty, 0)$ (b) $(-\infty, \infty) - \{0\}$
(c) $(-\infty, \infty)$ (d) $(0, \infty)$

One or More Than One Option(s) Correct Type

7. Let $f(x)=\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x \in R$ and $g(x)=\frac{\pi}{2}\sin x$ for all $x \in R$. Then which of the following is (are) true?

(a) Range of f is $\left[-\frac{1}{2},\frac{1}{2}\right]$
(b) Range of fog is $\left[-\frac{1}{2},\frac{1}{2}\right]$
(c) $\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{\pi}{6}$
(d) There is an $x \in R$ such that $(gof)(x) = 1$

8. Let $f(x) = \sin x$ and $g(x) = \ln|x|$. If the ranges of the composition functions fog and gof are R_1 and R_2 respectively, then

(a) $R_1 = \{u : -1 \le u < 1\}, R_2 = \{v : -\infty < v < 0\}$
(b) $R_1 = \{u : -\infty < u < 0\}, R_2 = \{v : -1 \le v \le 0\}$
(c) $R_1 = \{u : -1 \le u < 1\}, R_2 = \{v : -\infty < v < 0\}$
(d) $R_1 = \{u : -1 \le u \le 1\}, R_2 = \{v : -\infty < v \le 0\}$

9. Consider the following relations :

$R = \{(x, y) : x, y$ are real number and $x = wy$ for some rational number $w\}$

$$S = \left\{\frac{m}{n}, \frac{p}{q} : m, n, p, q \in Z \text{ such that } n, q \neq 0 \text{ and } qm = pn\right\},$$

then

(a) S is an equivalence relation but R is not an equivalence relation

(b) R and S both are equivalence relations.

(c) R is an equivalence relation but S is not an equivalence relation

(d) neither R nor S is an equivalence relation

10. $f(x) = \cos^2 x + \cos^2\left(\frac{\pi}{3}+x\right) - \cos x \cos\left(\frac{\pi}{3}+x\right)$ is

(a) an odd function (b) an even function

(c) a periodic function (d) $f(0) = f(1)$

11. Domain of $f(x) = \sin^{-1}[2 - 4x^2]$ is ([·] denotes the greatest integer function)

(a) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right] - \{0\}$ (b) $\left[-\frac{3}{2}, 0\right)$

(c) $\left[-\frac{3}{2}, 0\right) \cup \left(0, \frac{\sqrt{3}}{2}\right]$ (d) $\left[-\frac{\sqrt{3}}{2}, 8\right]$

12. If $f(x_1) - f(x_2) = f\left(\frac{x_1 - x_2}{1 - x_1 x_2}\right)$ for $x_1, x_2 \in [-1, 1]$, then $f(x)$ is

(a) $\log\frac{(1-x)}{(1+x)}$ (b) $\tan^{-1}\frac{(1-x)}{(1+x)}$

(c) $\log\frac{(1+x)}{(1-x)}$ (d) $\tan^{-1}\frac{(1+x)}{(1-x)}$

13. S is a relation over the set R of all real numbers and it is given by $(a, b) \in S \Leftrightarrow ab \geq 0$. The, S is

(a) symmetric (b) reflexive

(c) transitive (d) all of these

Comprehension Type

Let us consider two non empty sets X and Y such that $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $Y = \{1, 2, 3, 4, 5, 6\}$.

14. The number of into functions from X to Y is

(a) $7^6 - 3.7!$ (b) $6^7 - 3.7!$

(c) $6^7 - 7!$ (d) $6^6 - 3.6!$

15. The number of many one functions from X to Y is

(a) 6^7 (b) 7^6 (c) $7^6 - 6^7$ (d) $6^7 - 6!$

Matrix Match Type

16. Match the following :

	Column I		Column II
P.	Let $f : R \to R$ satisfies $f(x + y) + f(x - y) = 2f(x)f(y) \forall x, y \in R$ and $f(0) \neq 0$. Then $f(x)$ is	1.	even
Q.	Let $f : R \to R$ is defined by $f(x) = \frac{e^{\lvert x \rvert} - e^{-x}}{e^x + e^{-x}}$, then $f(x)$ is	2.	odd
R.	Let $f : R \to R$ is defined by $f(x) = 2x + \sin x$, then $f(x)$ is	3.	into
		4.	many-one
		5.	bijective

	P	Q	R
(a)	1,4	3,4	2,5
(b)	1,4	3,2	4,5
(c)	1,2	2,4	3,5
(d)	1,2	3,4	2,5

Numerical Value Type

17. The function $f(x) = \lambda|\sin x| + \lambda^2|\cos x| + g(\lambda)$ has period equal to $\pi/2$, then λ is _____.

18. The number of equivalence relations that can be defined on set $\{a, b, c\}$, is _____.

19. Let $f(x) = \begin{cases} 1 + [x] & , \ x < -2 \\ |x| & , \ x \geq -2 \end{cases}$ (where [·] denotes the greatest integer function), then $f(f(-2.6))$ is _____.

20. Let f be a real-valued invertible function such that $f\left(\frac{2x-3}{x-2}\right) = 5x - 3, x \neq 2$. Then the value of $f^{-1}(12)$ is _____.

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……

No. of questions correct ……

Marks scored in percentage ……

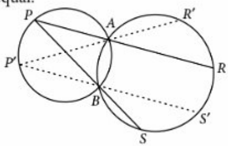
Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

OLYMPIAD CORNER

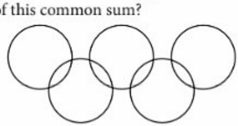
1. We are considering triangles ABC in space.
   (a) What conditions must be fulfilled by the angles α, β, γ of triangle ABC in order that there exists a point P in space such that $\angle APB$, $\angle BPC$, $\angle CPA$ are right angles?
   (b) Let d be the maximum distance among PA, PB, PC and let h be the longest altitude of triangle ABC. Show that $(\sqrt{6}/3)h \le d \le h$.

2. Two circles intersect at A and B. P is any point on an arc AB of one circle. The lines PA, PB intersect the other circle at R and S, as shown below. If P' is any other point on the same arc of the first circle and if R', S' are the points in which the lines $P'A$, $P'B$ intersect the other circle, prove that the arcs RS and $R'S'$ are equal.



3. For any positive integer n, evaluate a_n/b_n, where
$$a_n = \sum_{k=1}^{n} \tan^2 \frac{k\pi}{2n+1}, b_n = \prod_{k=1}^{n} \tan^2 \frac{k\pi}{2n+1}.$$

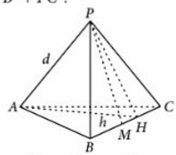
4. There are 9 regions inside the 5 rings of the Olympics. Put a different whole number from 1 to 9 in each so that the sum of the numbers in each ring is the same. What are the largest and the smallest values of this common sum?



5. Given are real numbers $a_1, a_2, ..., a_n$ with $\sum_{i=1}^{n} a_i = 0$. Determine
$$\sum_{i=1}^{n} \frac{1}{a_i(a_i + a_{i+1})(a_i + a_{i+1} + a_{i+2})...(a_i + a_{i+1} + \cdots + a_{i+n-2})}$$
where $a_{n+1} = a_1$, $a_{n+2} = a_2$, etc., assuming that the denominators are non zero.

SOLUTIONS

1. (a) By Pythagoras theorem, we have $AB^2 = PA^2 + PB^2$, $AC^2 = PA^2 + PC^2$ and $BC^2 = PB^2 + PC^2$.



Hence $AB^2 + AC^2 - BC^2 = 2PA^2 > 0$, thus $\angle BAC < 90°$, *i.e.*, $\alpha < 90°$.

Similarly, we get $\beta < 90°$ and $\gamma < 90°$. Conversely, if α, β and γ are all acute angles, we may prove that there exists a point P such that $\angle APB$, $\angle ABC$, $\angle CPA$ are right angles.

(b) We may without loss of generality assume that $PA \ge PB \ge PC$, so $PA = d$, Because $AB^2 = AP^2 + BP^2 \ge AP^2 + CP^2 = AC^2$ and $AC^2 = AP^2 + CP^2 \ge BP^2 + CP^2 = BC^2$, we have $AB \ge AC \ge BC$.

Let H be the foot of the perpendicular from A to BC, then AH is the longest altitude of ΔABC, so $AH = h$. As $AP \perp BP$ and $AP \perp CP$, AP is perpendicular to the plane of BPC. Thus $AP \perp BC$ and $AP \perp PH$ so that $AP < AH$, *i.e.*, $d < h$. ...(1)

Because $AP \perp BP$ and $AH \perp BC$, we get BC is perpendicular to the plane of APH. Thus, we have $BC \perp PH$.

Let M be the midpoint of BC, then $PH \le PM$. As $\angle BPC = 90°$, we have $PM = BM = MC = \frac{1}{2}BC$. Hence, $2PH \le 2PM = BC$, so that

$$4PH^2 \le BC^2 = PB^2 + PC^2 \le 2PA^2 \quad ...(2)$$

As $\angle APH = 90°$, we get

$$PH^2 = AH^2 - AP^2 = h^2 - d^2 \quad ...(3)$$

From (2) and (3), we have

$$4(h^2 - d^2) \le 2d^2, \text{ i.e., } 2h^2 \le 3d^2,$$

from which, we have

$$\frac{\sqrt{6}}{3}h \le d. \quad ...(4)$$

From (1) and (4) we obtain $\frac{\sqrt{6}}{3}h \le d < h$, as required.

2. Because opposite angles of a cyclic quadrilateral are supplementary, we have that $\angle PBA = \pi - \angle ABS = \angle ARS$. Similarly $\angle PAB = \angle BSR$. Thus ΔPAB and ΔPSR are similar, from which

$$\frac{PA}{PS} = \frac{PB}{PR} = \frac{AB}{RS}$$

(Notice that this also gives the 'power of the point' result for P, $PA \cdot PR = PB \cdot PS$)

Similarly $\frac{P'A}{P'S} = \frac{P'B}{P'R'} = \frac{AB}{R'S'}$

Consider now traingles APS and $AP'S'$. We have $\angle APS = \angle APB = \angle AP'B = \angle AP'S'$, because P, P' lie on the same arc of chord AB of the one circle. From the fact that S and S' lie on the same arc of chord AB of the second circle $\angle AS'P' = \angle ASP$. But then ΔAPS and $\Delta AP'S'$ are similar. Thus

$$\frac{PA}{PS} = \frac{P'A}{P'S'}. \text{ So } \frac{AB}{RS} = \frac{AB}{R'S'}$$

Thus $RS = R'S'$ and the arcs are equal.

3. Using De Moivre's theorem

$$\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n,$$

one finds easily that

$$\sin n\theta = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}\theta \sin^{2k+1}\theta$$

So

$$\sin(2n+1)\theta = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \cos^{2n-2k}\theta \sin^{2k+1}\theta$$

$$= \tan\theta \cos^{2n+1}\theta \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \tan^{2k}\theta.$$

Thus

$$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \tan^{2k}\theta = 0$$

for $\theta = \frac{j\pi}{2n+1}, 1 \le j \le n$

So $\tan^2 \frac{j\pi}{2n+1}, 1 \le j \le n$, are the roots of

$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} x^k = 0$ and thus also of

$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} x^{n-k} = 0.$ Since a_n and b_n are the sum and product of the roots, respectively, we have

$$a_n = \binom{2n+1}{2} = n(2n+1) \text{ and } b_n = \binom{2n+1}{2n} = 2n+1,$$

and so $\frac{a_n}{b_n} = n$

4. For the five rings, we have

$$a + b = b + c + d = d + e + f = f + g + h = h + i = N. \quad ...(1)$$

Since we are dealing with the nine non-zero decimal digits, we have $\sum_{1}^{9} j = 9(10)/2 = 45$. The five regions sum to a common N for $45/5 = 9$ but then one pair must be $9 + 0$ or one triplet $9 + 0 + 0$, which isn't allowed. So $N > 9$. Since $a + b = h + i$, there must be at least two pairs of decimal digits that sum to N. For $10 \le N \le 15$, we have

$$N = 9 + a = 8 + (1 + a) = ..., \text{ for } 1 \le a \le 6$$

while $N = 16 = 9 + 17$ and $N = 17 = 9 + 8$ only. So $N \le 15$.

From (1)

$$a + b = b + c + d \text{ or } a = c + d \quad ...(2)$$

and

$$h + i = f + g + h \text{ or } i = f + g \quad ...(3)$$

The five central digits must equal $45 - 2N$

$$(c + d) + e + (f + g) = a + e + i = 45 - 2N$$

So, we have

N	$2N$	$45 - 2N$	a, e, i
10	20	25	9, 8 – no digit available
11	22	23	9, 8, 6;...
12	24	21	9, 8, 4;...
13	26	19	9, 8, 2;...
14	28	17	9, 7, 1;...
15	30	15	9, 5, 1;...

So $11 \le N \le 15$.

5. Let

$$S=\sum_{i=1}^{n}\frac{1}{a_i\left(a_i+a_{i+1}\right)\ldots\left(a_i+a_{i+1}+\cdots+a_{i+n-2}\right)}$$

$$=\sum_{i=1}^{n}\frac{1}{a_{i+1}\left(a_{i+1}+a_{i+2}\right)\ldots\left(a_{i+1}+a_{i+2}+\cdots+a_{i+n-1}\right)}$$

be the given sum. Let $b_i=\sum_{r=1}^{i}a_r$ so $b_n = 0$. Also, let $p=\prod_{r=1}^{n-1}b_r$. Since for all $j \ne i$

$$b_j - b_i = b_n - b_i + b_j = a_{i+1} + a_{i+2} + \cdots + a_n + a_1 + a_2 + \cdots + a_j$$

$$= a_{i+1} + a_{i+2} + \cdots + a_n + a_{n+1} + \cdots + a_{n+j},$$

We have

$$S=\sum_{i=1}^{n}\prod_{\substack{1\le j\le n\\ j\ne i}}\frac{1}{b_j-b_i}=\frac{1}{p}+\sum_{i=1}^{n-1}\prod_{\substack{1\le j\le n\\ j\ne i}}\frac{1}{b_j-b_i}$$

$$=\frac{1}{p}+\sum_{i=1}^{n-1}\left(-\frac{1}{b_i}\prod_{\substack{1\le j\le n-1\\ j\ne i}}\frac{1}{b_j-b_i}\right)$$

Using Lagrange's interpolation, we let

$$F(x)=\sum_{i=1}^{n-1}\left(-\frac{1}{p}\prod_{\substack{1\le j\le n-1\\ j\ne i}}\frac{b_j-x}{b_j-b_i}\right)$$

So that $F(b_k) = -1/p$ for all $1 \le k \le n - 1$. Note that $F(x)$ is a polynomial in x of degree at most $n - 2$, yet it is a constant at $n - 1$ distinct values. Hence $F(x)$ is a constant, $F(x) = -1/p$. Since

$$-\frac{1}{p}\prod_{\substack{1\le j\le n-1\\ j\ne i}}\frac{b_j}{b_j-b_i}=-\frac{1}{b_i}\prod_{\substack{1\le j\le n-1\\ j\ne i}}\frac{1}{b_j-b_i},$$

we get $F(0) = S - 1/p$ and thus

$$S=F(0)+\frac{1}{p}=0.$$

◆◆

QUANTITATIVE APTITUDE

Useful for Bank PO, Specialist Officers & Clerical Cadre, BCA, MAT, CSAT, CDS and other such examinations

1. The maximum number of students among them 1001 pens and 910 pencils can be distributed in such a way that each student gets the same number of pens and same number of pencils is

   (a) 91 (b) 910 (c) 1001 (d) 1911

2. Each boy contributed rupees equal to the number of girls and each girl contributed rupees equal to the number of boys in a class of 60 students. If the total contribution thus collected is ₹1600, how many boys are there in the class?

   (a) 25 (b) 30
   (c) 50 (d) Data inadequate

3. The average temperature of the town in the first four days of a month was 58 degrees. The average for the 2^{nd}, 3^{rd}, 4^{th} and 5^{th} days was 60 degrees. If the temperature of the 1^{st} and 5^{th} days were in the ratio 7 : 8, then what is the temperature on the 5^{th} day?

   (a) 64 degrees (b) 62 degress
   (c) 56 degrees (d) None of these

4. The sum of four numbers is 64. If you add 3 to the first number, 3 is subtracted from the second number, the third is multiplied by 3 and the fourth is divided by 3, then all the results are equal. What is the difference between the largest and the smallest of the original numbers?

   (a) 21 (b) 27
   (c) 32 (d) None of these

5. A spider climbed $62\frac{1}{2}\%$ of the height of the pole in one hour and in the next hour it covered $12\frac{1}{2}\%$ of the remaining height. If the height of the pole is 192 m, then distance climbed in second hour is

   (a) 3 m (b) 5 m (c) 7 m (d) 9 m

6. Samant bought a microwave oven and paid 10% less than the original price. He sold it with 30% profit on the price he had paid. What percentage of profit did Samant earn on the original price?

   (a) 17% (b) 20% (c) 27% (d) 32%

7. The electricity bill of a certain establishment is partly fixed and partly varies as the number of units of electricity consumed. When in a certain month 540 units are consumed, the bill is ₹1800. In another month 620 units are consumed and the bill is ₹2040. In yet another month 500 units are consumed. The bill for that month would be

   (a) ₹1560 (b) ₹1840 (c) ₹1680 (d) ₹1950

8. Which of the following statements is/are necessary to answer the given question.

   Three friends, P, Q and R started a partnership business investing money in the ratio of 5 : 4 : 2 respectively for a period of 3 years. What is the amount received by P as his share in the total profit?

   Statement (I) : Total amount invested in the business in ₹22,000.

   Statement (II) : Profit earned at the end of 3 years is $\frac{3}{8}$ of the total investment.

   Statement (III) : The average amount of profit earned per year is ₹2750.

   (a) Statement (I) or Statement (II) or Statement (III)
   (b) Either Statement (III) only, or Statement (I) and Statement (II) together
   (c) Any two of the three
   (d) All Statement (I), Statement (II) and Statement (III)

9. Ronald and Elan are working on an assignment. Ronald takes 6 hours to type 32 pages on a computer, while Elan takes 5 hours to type 40 pages. How much time will they take, working together on two different computers to type an assignment of 110 pages?
(a) 7 hours 30 minutes
(b) 8 hours
(c) 8 hours 15 minutes
(d) 8 hours 25 minutes

10. A person borrowed ₹500 @ 3% per annum S.I. and ₹600 @ $4\frac{1}{2}$% per annum on the agreement that the whole sum will be returned only when the total interest becomes ₹126. The number of years, after which the borrowed sum is to be returned is
(a) 2 (b) 3
(c) 4 (d) 5

11. A park square in shape has a 3 metre wide road inside it running along its sides. The area occupied by the road is 1764 square metres. What is the perimeter along the outer edge of the road ?
(a) 576 metres (b) 600 metres
(c) 640 metres (d) Data inadequate

12. Two metallic right circular cones having their heights 4.1 cm and 4.3 cm and the radii of their bases 2.1 cm each, have been melted together and recast into a sphere. Find the diameter of the sphere.
(a) 7.1 cm (b) 4.2 cm
(c) 3.1 cm (d) 6.4 cm

13. In a class, 30% of the students offered English, 20% offered Hindi and 10% offered both. If a student is selected at random, what is the probability that he has offered English or Hindi ?
(a) $\frac{2}{5}$ (b) $\frac{3}{4}$
(c) $\frac{3}{5}$ (d) $\frac{3}{10}$

Direction (14-16) : Study the table and answer the given questions.

Expenditures of a company (in Lakh rupees) per annum over the given years.

Items of expenditure / Years	Salary	Fuel and Transport	Bonus	Interest on Loans	Taxes
1998	288	98	3.00	23.4	83
1999	342	112	2.52	32.5	108
2000	324	101	3.84	41.6	74
2001	336	133	3.68	36.4	88
2002	420	142	3.96	49.4	98

14. The ratio between the total expenditure on Taxes for all the years and the total expenditure on Fuel and Transport for all the years respectively is approximately.
(a) 4 : 7 (b) 10 : 13
(c) 15 : 18 (d) 5 : 8

15. What is the average amount of interest per year which the Company had to pay during this period?
(a) ₹32.43 lakhs (b) ₹33.72 lakhs
(c) ₹34.18 lakhs (d) ₹36.66 lakhs

16. Total expenditure on all these items in 1998 was approximately what percent of the total expenditure in 2002 ?
(a) 62% (b) 66% (c) 69% (d) 71%

17. A booster pump can be used for filling as well as for emptying a tank. The capacity of the tank is 2400 m^3. The emptying capacity of the tank is 10 m^3 per minute higher than its filling capacity and the pump needs 8 minutes lesser to empty the tank than it needs to fill it. What is the filling capacity of the pump ?
(a) 50 m^3/min. (b) 60 m^3/min.
(c) 72 m^3/min. (d) None of these

18. The number of students in each section of a school is 24. After admitting new students, three new sections were started. Now, the total number of sections is 16 and there are 21 students in each section. The number of new students admitted is
(a) 14 (b) 24 (c) 48 (d) 114

19. 4 mat-weavers can weave 4 mats in 4 days. At the same rate, how many mats would be woven by 8 mat-weavers in 8 days ?
(a) 4 (b) 8 (c) 12 (d) 16

20. In dividing a number by 585, a student employed the method of short division. He divided the number successively by 5, 9 and 13 (factors of 585) and got the remainders 4, 8, 12 respectively. If he had divided the number by 585, the remainder would have been

(a) 24 (b) 144 (c) 292 (d) 584

SOLUTIONS

1. **(a) :** Required number of students = H.C.F. of 1001, and 910 = 91.

2. **(d):** Let number of boys = x
Then, number of girls = $(60 - x)$
$\therefore\ x(60 - x) + (60 - x)\,x = 1600$
$\Rightarrow 2x^2 - 120x + 1600 = 0 \Rightarrow x^2 - 60x + 800 = 0$
$\Rightarrow (x - 40)(x - 20) = 0 \Rightarrow x = 40$ or $x = 20$
Hence, data is inadequate.

3. **(a) :** Sum of temperature on 1^{st}, 2^{nd}, 3^{rd} and 4^{th} days $= (58 \times 4) = 232$ degrees ...(i)
Sum of temperature on 2^{nd}, 3^{rd}, 4^{th} and 5^{th} days $= (60 \times 4) = 240$ degrees ...(ii)
Subtracting (i) from (ii), we get
Temp. on 5^{th} day – Temp. on 1^{st} day = 8 degrees
Let the temperature on 1^{st} and 5^{th} days be $7x$ and $8x$ degrees respectively.
Then, $8x - 7x = 8 \Rightarrow x = 8$
$\therefore$ Temperature on the 5^{th} day $= 8x = 64$ degrees.

4. **(c) :** Let the four numbers be A, B, C and D. Let
$A + 3 = B - 3 = 3C = \frac{D}{3} = x$
Then, $A = x - 3, B = x + 3, C = \frac{x}{3}$ and $D = 3x$
$A + B + C + D = 64 \Rightarrow (x - 3) + (x + 3) + \frac{x}{3} + 3x = 64$
$\Rightarrow 16x = 192 \Rightarrow x = 12$
Thus, the numbers are 9, 15, 4 and 36
$\therefore$ Required difference = (36 – 4) = 32.

5. **(d):** Height climbed in second hour $= 12\frac{1}{2}\%$ of $\left(100 - 62\frac{1}{2}\right)\%$ of 192 m
$= \left(\frac{25}{2} \times \frac{1}{100} \times \frac{75}{2} \times \frac{1}{100} \times 192\right) \text{m} = 9 \text{ m}$

6. **(a) :** Let original price = ₹ 100
Then, C.P. = ₹ 90
S.P. = 130% of ₹ 90 = ₹ $\left(\frac{130}{100} \times 90\right)$ = ₹ 117
$\therefore$ Required percentage = (117 – 100)% = 17%

7. **(c) :** Let the fixed amount be ₹ x and the cost of each unit be ₹ y. Then,
$540y + x = 1800$ ...(i)
$620y + x = 2040$ ...(ii)
On subtracting (i) from (ii), we get
$80y = 240 \Rightarrow y = 3$
Putting $y = 3$ in (i), we get
$540 \times 3 + x = 1800 \Rightarrow x = 180$
$\therefore$ Fixed amount = ₹ 180, Cost per unit = ₹ 3
Total cost for consuming 500 units
= ₹ (180 + 500 × 3) = ₹ 1680.

8. **(b):** Statement (I) and Statement (II) give, profit after 3 years = ₹ $\left(\frac{3}{8} \times 22000\right)$ = ₹ 8250
From Statement (III) also, profit after 3 years
= ₹ (2750 × 3) = ₹ 8250
$\therefore$ P's share = ₹ $\left(8250 \times \frac{5}{11}\right)$ = ₹ 3750
Thus, (either Statement (III) is redundant) or (Statement (I) and Statement (II) are redundant).
$\therefore$ Correct answer is (b).

9. **(c) :** Number of pages typed by Ronald in 1 hour
$= \frac{32}{6} = \frac{16}{3}$
Number of pages typed by Elan in 1 hour $= \frac{40}{5} = 8$
Number of pages typed by both in 1 hour
$= \left(\frac{16}{3} + 8\right) = \frac{40}{3}$
$\therefore$ Time taken by both to type 110 pages
$= \left(110 \times \frac{3}{40}\right)$ hours $= 8\frac{1}{4}$ hours = 8 hours 15 minutes.

10. **(b):** Let the time be x years
Then, $\left(\frac{500 \times 3 \times x}{100}\right) + \left(\frac{600 \times 9 \times x}{100 \times 2}\right) = 126$
$\Rightarrow 15x + 27x = 126 \Rightarrow x = 3$
$\therefore$ Required time = 3 years.

11. **(b):** Let the length of the outer edge be x m.
Then length of the inner edge = $(x - 6)$ m
$\therefore\ x^2 - (x - 6)^2 = 1764 \Rightarrow x^2 - (x^2 - 12x + 36) = 1764$
$\Rightarrow 12x = 1800 \Rightarrow x = 150$
$\therefore$ Required perimeter = $(4x)$ m = (4×150) m = 600 m.

12. (b): Volume of sphere = Volume of 2 cones

$$= \left[\frac{1}{3}\pi \times (2.1)^2 \times 4.1 + \frac{1}{3}\pi \times (2.1)^2 \times 4.3\right] \text{cm}^3$$

$$= \frac{1}{3}\pi \times (2.1)^2 (8.4) \text{cm}^3$$

Let the radius of the sphere be R.

$$\therefore \quad \frac{4}{3}\pi R^3 = \frac{1}{3}\pi (2.1)^3 \times 4$$

$\Rightarrow R = 2.1$ cm

Hence, diameter of the sphere = 4.2 cm.

13. (a): $P(\text{E}) = \frac{30}{100} = \frac{3}{10}$, $P(\text{H}) = \frac{20}{100} = \frac{1}{5}$

and $P(\text{E} \cap \text{H}) = \frac{10}{100} = \frac{1}{10}$

P (E or H) = P (E ∪ H)

$= P(\text{E}) + P(\text{H}) - P(\text{E} \cap \text{H})$

$$= \left(\frac{3}{10} + \frac{1}{5} - \frac{1}{10}\right) = \frac{4}{10} = \frac{2}{5}.$$

14. (b): Required ratio $= \frac{(83+108+74+88+98)}{(98+112+101+133+142)}$

$$= \frac{451}{586} \approx \frac{1}{1.3} = 10:13$$

15. (d): Average amount of interest paid by the company during the given period

$$= ₹\left(\frac{23.4+32.5+41.6+36.4+49.4}{5}\right) \text{lakhs}$$

$$= ₹\left(\frac{183.3}{5}\right) \text{lakhs} = ₹\ 36.66 \text{ lakhs}$$

16. (c): Required percentage

$$= \left[\frac{288+98+3.00+23.4+83}{420+142+3.96+49.4+98} \times 100\right]\%$$

$$= \left(\frac{495.4}{713.36} \times 100\right)\% \approx 69.45\% \approx 69\% \text{ (approx.)}$$

17. (a): Let the filling capacity of the pump be x m^3/min. Then, emptying capacity of the pump

$= (x + 10)$ m^3/min

According to question, $\frac{2400}{x} - \frac{2400}{(x+10)} = 8$

$\Rightarrow x^2 + 10x - 3000 = 0$

$(x - 50)(x + 60) = 0 \quad \Rightarrow \quad x = 50$

[neglecting the – ve value of x]

So, filling capacity of the pump = 50 m^3/min.

18. (b): Original number of sections = (16 – 3) = 13

Original number of students = (24 × 13) = 312

Present number of students = (21 × 16) = 336

∴ Number of new students admitted = (336 – 312)

= 24

19. (d): Let the required number of mats be x.

More weavers, More mats (Direct Proportion)

More days, More mats (Direct Proportion)

$$\left.\begin{array}{ll}\text{Weavers} & 4:8 \\ \text{Days} & 4:8\end{array}\right\} :: 4:x$$

$$\therefore \quad 4 \times 4 \times x = 8 \times 8 \times 4 \quad \Rightarrow x = \frac{8 \times 8 \times 4}{(4 \times 4)} = 16$$

So, the required number of mats = 16.

20. (d): $z = 13 \times 1 + 12 = 25$,

$y = 9 \times z + 8 = 9 \times 25 + 8 = 233$,

$x = 5 \times y + 4 = 5 \times 233 + 4 = 1169$

5	x
9	$y - 4$
13	$z - 8$
	$1 - 12$

Hence, on dividing 1169 by 585, remainder = 584.

YOU ASK WE ANSWER

Do you have a question that you just can't get answered?

Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

1. Prove that

$${}^nC_3 + {}^nC_7 + {}^nC_{11} + ... = \frac{1}{2}\left\{2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right\}$$

– Chinmay, Assam

Ans. $\because$ In given series difference in suffixes is 4.

Now $(1)^{1/4} = (\cos 0 + i\sin 0)^{1/4}$

$= (\cos 2r\pi + i\sin 2r\pi)^{1/4}$

$= \cos\frac{r\pi}{2} + i\sin\frac{r\pi}{2}$, where $r = 0, 1, 2, 3$

Four roots of unity are $1, i, -1$ and $-i$

$= 1, \alpha, \alpha^2$ and α^3 (say)

Also $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$

Putting $x = 1, \alpha, \alpha^2$ and α^3, we get

$2^n = \sum_{r=0}^{n} {}^nC_r$ ...(1), $(1+\alpha)^n = \sum_{r=0}^{n} {}^nC_r\alpha^r$ ...(2)

$(1+\alpha^2)^n = \sum_{r=0}^{n} {}^nC_r\alpha^{2r}$ ...(3)

and $(1+\alpha^3)^n = \sum_{r=0}^{n} {}^nC_r\alpha^{3r}$ ...(4)

Multiplying (1) by 1, (2) by α, (3) by α^2 and (4) by α^3 and adding, we get

$2^n + \alpha(1+\alpha)^n + \alpha^2(1+\alpha^2)^n + \alpha^3(1+\alpha^3)^n$

$= \sum_{r=0}^{n} {}^nC_r\left(1+\alpha^{r+1}+\alpha^{2r+2}+\alpha^{3r+3}\right)$ ...(5)

For $r = 3, 7, 11, ...$ R.H.S. of (5) is

${}^nC_3(1+\alpha^4+\alpha^8+\alpha^{12}) + {}^nC_7(1+\alpha^8+\alpha^{16}+\alpha^{24})$

$+ {}^nC_{11}(1+\alpha^{12}+\alpha^{24}+\alpha^{36}) + ...$

$= 4({}^nC_3 + {}^nC_7 + {}^nC_{11} + ...)$ $(\because \alpha^4 = 1)$

and L.H.S. of (5)

$= 2^n + i(1+i)^n + i^2(1+i^2)^n + i^3(1+i^3)^n$

$= 2^n + i\{(1+i)^n - (1-i)^n\} = 2^n - 2^{n/2}\cdot 2\sin\frac{n\pi}{4}$

$\therefore 4\left({}^nC_3 + {}^nC_7 + {}^nC_{11} + ...\right) = 2\left(2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right)$

$\Rightarrow {}^nC_3 + {}^nC_7 + {}^nC_{11} + ... = \frac{1}{2}\left(2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right)$

2. Prove that $\frac{3}{4} \le \sin^2\theta + \cos^4\theta \le 1$ for all real θ.

– Nandini, Gujarat

Ans. $\sin^2\theta + \cos^4\theta = \cos^4\theta - \cos^2\theta + 1$

$= \left\{(\cos^2\theta)^2 - 2.\frac{1}{2}\cos^2\theta + \frac{1}{4}\right\} + \frac{3}{4}$

$= \left(\cos^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4} \ge \frac{3}{4}$

Also, $\sin^2\theta + \cos^4\theta = \cos^4\theta - \cos^2\theta + 1$

$= \cos^2\theta(\cos^2\theta - 1) + 1 = 1 - \sin^2\theta\cos^2\theta \le 1$

$\therefore \frac{3}{4} \le \sin^2\theta + \cos^4\theta \le 1.$

3. Evaluate $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx$

– Raman, A.P.

Ans. Let

$$I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx = \int \frac{\frac{\pi}{2} - 2\cos^{-1}\sqrt{x}}{\frac{\pi}{2}}dx$$

$$= \int\left(1 - \frac{4}{\pi}\cos^{-1}\sqrt{x}\right)dx = x - \frac{4}{\pi}\int 1\cdot\cos^{-1}\sqrt{x}\,dx$$

$$= x - \frac{4}{\pi}\left[\cos^{-1}\sqrt{x}\cdot x - \int x\cdot\frac{-1}{\sqrt{1-(\sqrt{x})^2}}\cdot\frac{1}{2\sqrt{x}}dx\right]$$

$$\therefore I = x - \frac{4x}{\pi}\cos^{-1}\sqrt{x} - \frac{2}{\pi}\int\frac{\sqrt{x}}{\sqrt{1-x}}dx$$

Put $x = \cos^2\theta$; then $dx = -2\cos\theta\cdot\sin\theta\, d\theta$

$$\therefore \int\frac{\sqrt{x}}{\sqrt{1-x}}dx = \int\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}(-2\cos\theta\cdot\sin\theta)d\theta$$

$$= -\int 2\cos^2\theta\, d\theta = -\int(1+\cos 2\theta)d\theta$$

$$= -\left\{\theta + \frac{\sin 2\theta}{2}\right\} + c = -\cos^{-1}\sqrt{x} - \sqrt{x}.\sqrt{1-x} + c$$

$$\therefore I = x - \frac{4x}{\pi}\cos^{-1}\sqrt{x} - \frac{2}{\pi}\left\{-\cos^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x} + c\right\}$$

$$= x + \frac{2}{\pi}(1-2x)\cos^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x(1-x)} + c.$$

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